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Effect of Space Dimensions on Equilibrium Solutions of Cahn–Hilliard and Conservative Allen–Cahn Equations

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Abstract. In this study, we investigate the effect of space dimensions on the equilibrium solutions of the Cahn–Hilliard (CH) and conservative Allen–Cahn (CAC) equations in one, two, and three dimensions. The CH and CAC equations are fourth-order parabolic partial and second-order integro-partial differential equations, respectively. The former is used to model phase separation in binary mixtures, and the latter is used to model mean curvature flow with conserved mass. Both equations have been used for modeling various interface problems. To study the space-dimension effect on both the equations, we consider the equilibrium solution profiles for symmetric, radially symmetric, and spherically symmetric drop shapes. We highlight the different dynamics obtained from the CH and CAC equations. In particular, we find that there is a large difference between the solutions obtained from these equations in three-dimensional space.

AMS subject classifications: 65M55, 65N06, 76D05
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1. Introduction

In this study, we investigate the effect of space dimensions on the equilibrium solutions of the Cahn–Hilliard (CH) and conservative Allen–Cahn (CAC) equations in one, two, and three dimensions. The CH equation is given as

$$\frac{\partial \phi}{\partial t}(\mathbf{x},t) = \Delta \mu(\mathbf{x},t), \qquad \mathbf{x}\in\Omega, \quad t>0, \tag{1.1a}$$

$$\mu(\mathbf{x},t) = F'(\phi(\mathbf{x},t)) - \epsilon^2 \Delta\phi(\mathbf{x},t), \tag{1.1b}$$

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$$\mathbf{n}\cdot\nabla\phi(\mathbf{x},t)=\mathbf{n}\cdot\nabla\mu(\mathbf{x},t)=0,\qquad \mathbf{x}\in\partial\Omega,\quad t>0, \tag{1.1c}$$

where the order parameter $\phi(\mathbf{x},t)$ is the difference between two concentrations in a binary mixture in the domain $\Omega\subset\mathbb{R}^d$ $(d=1,2,3)$. $F(\phi)=0.25(\phi^2-1)^2$ is the free-energy density, ϵ is a positive constant related to the interfacial thickness, and $\mathbf{n}$ is the outward normal vector at the boundary. The CH equation was introduced to model phase separation phenomena in binary alloys [1]. A review paper [2] that documents the physical, mathematical, and numerical derivations of the CH equation is available. The CAC equation is given as

$$\frac{\partial\phi}{\partial t}(\mathbf{x},t)=-\frac{F'(\phi(\mathbf{x},t))}{\epsilon^2}+\Delta\phi(\mathbf{x},t)+\beta(t)\sqrt{F(\phi(\mathbf{x},t))},\quad \mathbf{x}\in\Omega,\quad t>0, \tag{1.2a}$$

$$\mathbf{n}\cdot\nabla\phi(\mathbf{x},t)=0,\qquad \mathbf{x}\in\partial\Omega,\quad t>0, \tag{1.2b}$$

where

$$\beta(t)=\int_\Omega F'(\phi(\mathbf{x},t))d\mathbf{x}\Big/\Big[\epsilon^2\int_\Omega\sqrt{F(\phi(\mathbf{x},t))}d\mathbf{x}\Big].$$

This equation is a second-order integro-partial differential equation, which is used for modeling mean curvature flows under the assumption of conservation of mass [3]. Both the equations and their modified forms can be used to model various interface problems in phase separation, fluids, topology optimization, tumor growth, image segmentation, phase field crystals, and so on [4–11]. In [12], the CH and CAC equations were compared as phase models, and the different dynamics between the two equations were specifically depicted. Please refer to [12] for further information.

The primary objective of this study is to investigate equilibrium solutions of the two equations in one, two, and three dimensions. In particular, we highlight the different dynamics of the CH and CAC equations.

The remainder of this paper is organized as follows: In Section 2, we describe the numerical solution algorithms of the CH and CAC equations. In Section 3, we present the numerical experiments. Finally, the conclusions are presented in Section 4.

2. Numerical solution

In this section, we consider numerical solutions in symmetric, radially symmetric, and spherically symmetric forms. The symmetric form of the CH equation (1.1a) and (1.1b) is as follows:

$$\phi_t(r,t)=\frac{1}{r^{d-1}}[r^{d-1}\mu_r(r,t)]_r,\qquad r\in\Omega,\quad t>0, \tag{2.1a}$$

$$\mu(r,t)=F'(\phi(r,t))-\frac{\epsilon^2}{r^{d-1}}[r^{d-1}\phi_r(r,t)]_r, \tag{2.1b}$$

where d is the space dimension. We discretize the CH equation (2.1a) and (2.1b) in $\Omega=(0,a)$. Let N_r be a positive even integer, $h=a/N_r$ be the uniform mesh size,