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Superconvergence of the Composite Rectangle Rule for Computing Hypersingular Integral on Interval

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Abstract. The generalized middle rectangle rule for the computation of certain hypersingular integrals is discussed. A generalized middle rectangle rule with the density function approximated and the singular kernel analysis calculated is presented and the asymptotic expansion of error functional is obtained. When the special function in the error functional equals to zero, the superconvergence point is obtained and the superconvergence phenomenon which is one order higher than the general case is presented. At last, numerical examples are given to confirm the theoretical results.

AMS subject classifications: 32A55, 40A10, 65D30

Key words: Hypersingular integral, middle rectangle rule, asymptotic expansion, superconvergence phenomenon.

1. Introduction

In the following, we consider the following integral

$$I_\alpha(f,y) := \neq\!\!\!\!\!\int_a^b \frac{f(x)}{|x-y|^{p+\alpha}}dx, \qquad y\in(a,b), \quad \alpha\in(0,1), \quad p=1,2, \tag{1.1}$$

where $\neq\!\!\!\!\int_a^b$ denotes a hypersingular integral and y is the singular point. There are lots of papers to investigate the case $\alpha=1$ is called hypersingular integral and $\alpha=0$ is called Cauchy principal value integral, such as Gauss formula [1, 2, 8–13, 30–32], Newton-Cotes methods [16, 17, 19–21, 23], cubic spline rule [28] and so on. In the following, we consider the hypersingular integral which have not been studied intensively.

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The definition of the hypersingular integral which is considered in Hadamard finite-part sense as,

$$\begin{aligned}&\,=\!\!\!\!\!\!\int_a^b \frac{f(x)}{|x-y|^{p+\alpha}}dx \\ &=\lim_{\varepsilon\to 0}\left\{\left(\int_a^{y-\varepsilon}+\int_{y+\varepsilon}^b\right)\frac{f(x)}{|x-y|^{p+\alpha}}dx-\frac{2f(y)}{(p+\alpha-1)\varepsilon^{p+\alpha-1}}\right\},\end{aligned} \tag{1.2}$$

where $0<\alpha<1$, if $f^{(p)}(x)$ is Holder continuous on $[a,b]$, $f(x)$ is said to be integrable in Hadamard finite-part sense.

Integrals of this kind usually appear in aerodynamics, wave propagation, fluid mechanics and so on [14, 15], relation to boundary integral equations [18] and finite-part integral equations [22].

We set a uniform partition $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ with its mesh size $h=(b-a)/n$ and

$$\hat{x}_i=x_i+h/2,\qquad i=0,1,2,\cdots,n-1. \tag{1.3}$$

We also define $f_C(x)$ to be the rectangle rule interpolation for $f(x)$ as

$$f_C(x)=f(\hat{x}_i),\qquad i=0,1,\cdots,n-1, \tag{1.4}$$

and a linear transformation

$$x=\hat{x}_i(\tau):=(\tau+1)\frac{x_{i+1}-x_i}{2}+x_i,\qquad i=0,1,\cdots,n-1,\quad \tau\in[-1,1], \tag{1.5}$$

from the reference element $[-1,1]$ to the subinterval $[x_i,x_{i+1}]$.

Replacing $f(x)$ in (1.1) with $f_C(x)$ gives the middle rectangle rule:

$$I_{n\alpha}(f,y):=\,=\!\!\!\!\!\!\int_a^b\frac{f_C(x)}{|x-y|^{p+\alpha}}dx=\sum_{i=0}^{n-1}\omega_{i\alpha}(y)f(\hat{x}_i)=I_\alpha(f,y)-E_{n\alpha}(f,y), \tag{1.6}$$

where $\omega_{i\alpha}(s)$ is the coefficients, $E_{n\alpha}(f,y)$ the error functional. For the case $\alpha=1$ hypersingular integral and $\alpha=0$ Cauchy principal value integral, the middle rectangle rule evaluate them have been studied in [3, 4] takes as

$$I_{n0}(f,y):=\,=\!\!\!\!\!\!\int_a^b\frac{f_C(x)}{x-y}dx=\sum_{i=0}^{n-1}\omega_{i0}(y)f(\hat{x}_i)=I_0(f,y)-E_{n0}(f,y), \tag{1.7a}$$

$$I_{n1}(f,y):=\,=\!\!\!\!\!\!\int_a^b\frac{f_C(x)}{(x-y)^2}dx=\sum_{i=0}^{n-1}\omega_{i1}(y)f(\hat{x}_i)=I_1(f,y)-E_{n1}(f,y). \tag{1.7b}$$

For $\alpha=0$, Liu [4] gave the following error estimate

$$I_0(f,y)-I_{n0}(f,y)=-2hf'(y)\ln\left(2\cos\frac{\tau\pi}{2}\right)+\mathcal{R}_{n0}(y), \tag{1.8}$$