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# NUMERICAL INVESTIGATION ON WEAK GALERKIN FINITE ELEMENTS

JUNPING WANG, XIU YE, AND SHANGYOU ZHANG

**Abstract.** The weak Galerkin (WG) finite element method is an effective and robust numerical technique for the approximate solution of partial differential equations. The essence of the method is the use of weak finite element functions and their weak derivatives computed with a framework that mimics the distribution or generalized functions. Weak functions and their weak derivatives can be constructed by using polynomials of arbitrary degrees; each chosen combination of polynomial subspaces generates a particular set of weak Galerkin finite elements in application to PDE solving. This article explores the computational performance of various weak Galerkin finite elements in terms of stability, convergence, and supercloseness when applied to the model Dirichlet boundary value problem for a second order elliptic equation. The numerical results are illustrated in 31 tables, which serve two purposes: (1) they provide detailed and specific guidance on the numerical performance of a large class of WG elements, and (2) the information shown in the tables may open new research projects for interested researchers as they interpret the results from their own perspectives.

**Key words.** Weak Galerkin, finite element methods, weak gradient, second-order elliptic problems, stabilizer-free.

## 1. Introduction

The weak Galerkin (WG) finite element method is an effective and robust numerical technique for the approximate solution of partial differential equations. It is a natural extension of the standard conforming Galerkin finite element method by substituting the classical derivatives with weakly defined discrete derivatives for discontinuous functions. The WG method was first introduced in [17, 18], and since then, it has been applied/extended to several classes of partial differential equations such as biharmonic equations, Stokes equations, Navier-Stokes equations, Brinkman equations, parabolic equations, Helmholtz equation, convection dominant problems, hyperbolic equations, and Maxwell's equations [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19].

The main idea of weak Galerkin finite element methods is the use of weak functions and their corresponding weak derivatives. For simplicity, we demonstrate the idea by using the second order elliptic problem that seeks an unknown function $u$ satisfying

$$-\nabla\cdot(\nabla u) = f \quad \text{in } \Omega, \tag{1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{2}$$

where $\Omega$ is a polygonal domain in $\mathbb{R}^2$. The primal weak form for the problem (1)-(2) seeks $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and satisfying

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{3}$$

For the variational problem (3), the weak functions possess the form of $v = \{v_0, v_b\}$ with $v = v_0$ inside of each element and $v = v_b$ on the boundary of the element. Both $v_0$ and $v_b$ can be approximated by polynomials in $P_\ell(T)$ and $P_s(e)$ respectively,

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where $T$ stands for an element and $e$ the edge of $T$, $\ell$ and $s$ are non-negative integers with possibly different values. Weak derivatives are defined for weak functions in the sense of distributions. Denote by $G_m(T)$ the vector space for weak gradient. Typical choices for $G_m(T)$ are $[P_m(T)]^d$ or the Raviart-Thomas elements $RT_m(T)$. Each particular combination of $(P_\ell(T), P_s(e), G_m(T))$ leads to a class of weak Galerkin finite element methods tailored for specific partial differential equations.

Weak Galerkin finite element methods have two forms for the problem (1)-(2). The first one is the standard formulation [11, 17] which seeks $u_h \in V_h$ such that $u_h = Q_b g$ on $\partial\Omega$ and satisfying

$$(\nabla_w u_h, \nabla_w v) + s(u_h, v) = (f, v) \quad \forall v \in V_h^0, \tag{4}$$

where $s(\cdot,\cdot)$ is a parameter independent stabilizer. Another one is the stabilizer-free formulation [1, 20, 21]: find $u_h \in V_h$ such that $u_h = Q_b g$ on $\partial\Omega$ and satisfying

$$(\nabla_w u_h, \nabla_w v) = (f, v) \quad \forall v \in V_h^0. \tag{5}$$

Removing the stabilizer from the original WG scheme simplifies the formulation and reduces the programming complexity arising from the stabilizer, but at the cost of increasing the computational complexity for the weak derivative. To have desired stability and convergence, stabilizer-free WG methods must use relatively high values of $m$ for approximating weak gradient in the WG element $(P_\ell(T), P_s(e), G_m(T))$.

The purpose of this paper is to investigate the performance of different WG elements computationally in the weak Galerkin finite element methods with or without stabilizers. The numerical results will be illustrated in 31 tables that are informative and clearly demonstrate special properties of each WG element. It should be noted that not all of the numerical phenomena shown in the tables have theoretical justifications in existing literature; researchers are encouraged to conduct a theoretical investigation for those of their interests.

Table 3 shows that the WG element $(P_k(T), P_k(e), [P_{k+1}]^2)$ has two orders of supercloseness in both energy and $L^2$ norms on rectangular partitions. Motivated by this computational result, we will provide a theoretical justification for this superconvergence in a forthcoming paper. We note that it has been proved in [2] that the WG element $(P_k(T), P_{k+1}(e), [P_{k+1}]^2)$ has two orders of supercloseness in both energy norm and $L^2$ norm, on general triangular meshes in Table 18. Furthermore, to overcome the poor performance of the WG element $(P_k(T), P_{k-1}(e), [P_{k+1}]^2)$ shown in Table 24 and 29, a new weak gradient was introduced in [22] so that the element can still converge in optimal order on general polytopal meshes.

The WG methods are designed for discontinuous approximations on general polytopal meshes. Due to limited space, this paper shall only consider the finite element partitions with rectangular and triangular elements.

## 2. Weak Galerkin Finite Element formulations

Let $\mathcal{T}_h$ be a partition of the domain $\Omega$ consisting of rectangles or triangles. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges or flat faces. For every element $T \in \mathcal{T}_h$, we denote by $h_T$ its diameter and mesh size $h = \max_{T\in\mathcal{T}_h} h_T$ for $\mathcal{T}_h$. Let $P_k(T)$ consist all the polynomials defined on $T$ of degree less or equal to $k$.

**Definition 1.** *For $T \in \mathcal{T}_h$ and $\ell, s \geq 0$, define a local WG element $W_{\ell,s}(T)$ as,*

$$W_{\ell,s}(T) = \{v = \{v_0, v_b\} : \ v_0|_T \in P_\ell(T), \ v_b|_e \in P_s(e), \ e \subset \partial T\}. \tag{6}$$