

# Numerical Analysis and Simulation of a Frictional Contact Problem with Wear, Damage and Long Memory

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**Abstract.** A frictional contact model accounting the wear of the contact surface caused by the friction and the mechanical damage of the material is considered. The deformable body is comprised of a viscoelastic material with long memory and the process is assumed to be quasistatic. The mechanical damage caused by tension or compression is included in the constitutive law and the damage function is modelled by a nonlinear parabolic inclusion. The wear is contained in the contact boundary conditions and wear function is modelled by a differential equation. Variational formulation of the model is governed by a coupled system consisting of a history-dependent variational inequality, a nonlinear parabolic variational inequality and an integral equation. A fully discrete scheme of the problem is studied and optimal error estimates are derived for the linear finite element method. Numerical simulations illustrate the model behaviour.

**AMS subject classifications:** 65M15, 65N21, 65N22

**Key words:** Variational inequality, damage, integral equation, numerical approximation, optimal order error estimate.

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## 1. Introduction

In this paper, we focus on a frictional contact problem between a viscoelastic body and a foundation. The effect due to the damage of the material, along with the wear caused by the friction, is considered. Frictional contact between a deformable body and a foundation is a kind of special phenomenon which occurs in diverse forms in various physical settings. In any particular case, the behavior of the body is affected by different factors, including the constitutive law of the body, the friction law describing the contact, the temperature influence and piezoelectricity effects. In particular, the reader can consult [3,5] for thermopiezoelectric materials contact problem. Therefore, various models have been developed in the contact mechanics. The decrease in the load-bearing capacity due to the appearance

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and growth of internal cracks have been spotted for a variety of materials such as concrete. A lot of efforts have been spent on the study of this problem since it exerts a deep influence on the life-span of the designed components and structures — cf. [1, 13, 22]. On the other hand, contact processes are often accompanied by material damage and the corresponding mathematical models along with related variational problems are established in [7, 15, 21]. Nevertheless, there are only a few numerical studies considering the effect of the internal damage on the contact processes — cf. [8, 9, 16]. As the contact process evolves, the contacting surfaces also evolve via their wear. Usually, sliding systems wear slowly but steadily. The corresponding models with wear have been studied in [20, 24].

In this work we consider numerical approximation of a system coupled by a general history-dependent variational inequality, a nonlinear parabolic variational inequality and an integral equation which models a quasistatic frictional contact problem with wear, damage and long memory. To the best of author's knowledge, this is the first paper devoted to numerical analysis of variational inequalities in contact problems with wear and damage. The model is nonstandard and since its variational formulation leads to two variational inequalities and an integral equation, the proofs of the corresponding results are technically complicated. We establish the existence and uniqueness result and use a numerical method for solving the system arising. Optimal error estimates for the scheme are derived under certain solution regularity assumptions.

We first study the following frictional contact problems. Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^d$ ,  $d = 2, 3$  occupied by a viscoelastic body. The boundary  $\Gamma$  of  $\Omega$  is assumed to be Lipschitz continuous and divided into three mutually disjoint parts  $\Gamma_D$ ,  $\Gamma_N$  and  $\Gamma_C$  such that the measure of  $\Gamma_D$ , denoted  $m(\Gamma_D)$ , is positive. Assume that the body is clamped on  $\Gamma_D$ , and the displacement field vanishes there. Time-dependent surface traction of density  $\mathbf{f}_N$  act on  $\Gamma_N$  and time-dependent volume forces of density  $\mathbf{f}_0$  act in  $\Omega$ . The evolutionary process of the mechanical state of the body is restricted to the time interval  $(0, T)$  with  $T > 0$ .

The notations  $\mathbf{u} = (u_i)$ ,  $\boldsymbol{\sigma} = (\sigma_{ij})$  and  $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$  are used for the displacement vector, the stress tensor and the linearised strain tensor, respectively. For the sake of simplicity, we do not indicate explicitly the dependence of variables on the spatial variable  $\mathbf{x}$ . The components of the linearised strain tensor  $\boldsymbol{\varepsilon}(\mathbf{u})$  are  $\varepsilon_{ij}(\mathbf{u}) = (1/2)(u_{i,j} + u_{j,i})$ , where  $u_{i,j} = \partial u_i / \partial x_j$ . The indices  $i, j, k, l$  run between 1 and  $d$  and, unless stated otherwise, the summation convention over repeated indices is used. An index following a comma indicates a partial derivative with respect to the corresponding component of the spatial variable  $\mathbf{x}$ . A superscript prime of a variable stands for the time derivative of the corresponding variable. The outward unit normal to  $\partial\Omega$  is denoted by  $\boldsymbol{\nu}$  and we write  $\nu_\nu$  and  $\boldsymbol{\nu}_\tau$  for the normal and tangential components of  $\boldsymbol{\nu}$  on  $\partial\Omega$ , i.e.  $\nu_\nu = \boldsymbol{\nu} \cdot \boldsymbol{\nu}$  and  $\boldsymbol{\nu}_\tau = \boldsymbol{\nu} - \nu_\nu \boldsymbol{\nu}$ . The normal and tangential components of the stress field  $\boldsymbol{\sigma}$  on the boundary are defined by  $\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}$  and  $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ , respectively. The symbol  $\mathbb{S}^d$  refers to the space of second order symmetric tensors on  $\mathbb{R}^d$ .

The mathematical model of the contact problem is stated as follows.

**Problem 1.1.** Find a displacement field  $\mathbf{u} : \Omega \times (0, T) \rightarrow \mathbb{R}^d$ , a stress field  $\boldsymbol{\sigma} : \Omega \times (0, T) \rightarrow \mathbb{S}^d$ , a damage field  $\zeta : \Omega \times (0, T) \rightarrow \mathbb{R}$  and a wear function  $w : \Gamma_C \times (0, T) \rightarrow \mathbb{R}_+$  such that