East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.100719.230320

Vol. **10**, No. 4, pp. 698-716
November 2020

ADMM-Based Methods for Nearness Skew-Symmetric and Symmetric Solutions of Matrix Equation $AXB = C$

Dongmei Yu[1,2], Cairong Chen[1,*] and Deren Han[1,3]

[1] *School of Mathematical Sciences, Beihang University, Beijing 100191, China.*
[2] *Institute of Optimization and Decision, Liaoning Technical University, Fuxin 123000, China.*
[3] *Beijing Advanced Innovation Center for Big Data and Brain Computing (BDBC), Beihang University, Beijing 100191, China.*

Received 10 July 2019; Accepted (in revised version) 23 March 2020.

Abstract. New equivalent forms of a matrix nearness problem are proposed and necessary and sufficient conditions for a skew-symmetric or symmetric matrix X^* to be a solution of the problem are obtained. In order to find approximate solutions of the problem, we employ two ADMM-based iterative methods and prove their global convergence. Numerical simulations demonstrate the effectiveness of the methods developed.

AMS subject classifications: 15A24, 15A39, 65F30

Key words: ADMM, matrix nearness problem, global convergence, preconditioning, iterative method.

1. Introduction

Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ real matrices, X^T the transpose of the matrix X, $\mathbb{SR}^{n\times n} := \{X|X^T = X, X \in \mathbb{R}^{n\times n}\}$ the set of all symmetric matrices, and $\mathbb{SSR}^{n\times n} = \{X|X^T = -X, X \in \mathbb{R}^{n\times n}\}$ the set of all skew-symmetric matrices. The set $\mathbb{R}^{m\times n}$ equipped with the inner product

$$\langle A,B\rangle = \mathrm{trac}(A^T B), \quad A,B \in \mathbb{R}^{m\times n}$$

and the Frobenius norm $\|A\| = \sqrt{\langle A,A\rangle}$, becomes a real Hilbert space. If $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times n$ matrices, then $A \circ B$ refers to their Hadamard product— i.e. $A \circ B := (a_{ij} b_{ij})$. We also write I_n for $n \times n$ identity matrix or simply I if its dimension is clear from the context. Besides, $\|\cdot\|_2$ denotes the Euclidean vector norm.

*Corresponding author. *Email addresses:* `yudongmei1113@163.com` (D. Yu), `cairongchen@buaa.edu.cn` (C. Chen), `handr@buaa.edu.cn` (D. Han)

Let $\bar{X} \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{m\times p}$ be given matrices. Here, we consider the matrix nearness problem

$$\begin{aligned} &\min_X \frac{1}{2}\left\|X - \bar{X}\right\|^2 \\ &\text{s.t.} \quad AXB = C, \\ &\qquad X \in \mathbb{SR}^{n\times n} \cup \mathbb{SSR}^{n\times n} \end{aligned} \tag{1.1}$$

under the assumption that the matrix equation $AXB = C$ is consistent.

The symmetric matrix nearness problem appears when considering linear systems with incomplete dates or revising dates [27], and to the best of our knowledge, the skew-symmetric matrix nearness problem is of theoretical interest as well.

The matrix nearness problem is related to projections in Hilbert spaces. Let $\mathbb{H}$ be a real Hilbert space with the norm $\|\cdot\|_{\mathbb{H}}$, $\mathbb{S}$ be a closed convex subset of $\mathbb{H}$ and $x \in \mathbb{H}$. The point in $\mathbb{S}$, nearest to x is called the projection of x onto $\mathbb{S}$ and is denoted by $\mathbb{P}_{\mathbb{S}}(x)$. More precisely, $\mathbb{P}_{\mathbb{S}}(x)$ is the solution of the following optimisation problem [2, 12]:

$$\|\mathbb{P}_{\mathbb{S}}(x) - x\|_{\mathbb{H}} = \min_{y\in\mathbb{S}} \|y - x\|_{\mathbb{H}}.$$

Now, we turn our attention to the matrix space $\mathbb{R}^{m\times n}$. The matrix nearness problem consists in finding a matrix X^*, which belongs to a constraint matrix set and is closest to a given matrix $\bar{X}$. This problem plays an important role in scientific computing, structure design, control theory and finite element model updating. The constraint matrix set mentioned is always the solution set (or the least square solution set) of certain matrix equations — cf. Refs. [10, 14] and the references therein. Since the preliminary estimation $\bar{X}$ is often obtained from experiments, it can lie outside of the constraint matrix set. Therefore, $\bar{X}$ has to be replaced by a nearness matrix X^* in the constraint matrix set used [17]. In recent years, the matrix nearness problem has been extensively studied — cf. Refs. [6,9,13,20–22, 24–27,29,30,36]. Huang *et al.* [18,19] determined a general solution of the problem (1.1) and used an iterative method to find skew-symmetric and optimal approximate solutions of the equation $AXB = C$.

This work is inspired by recent studies [27] concerning the nearness symmetric solutions of the equation $AXB = C$. The goal of this paper is twofold: to obtain new equivalent formulations of the matrix nearness problem (1.1) and, second, to develop new iterative methods for its solution. The matrix preconditioning technique provides an efficient solution procedure for various linear and nonlinear systems of equations — cf. [8, 28] and it is interesting to employ this approach to the matrix nearness problem (1.1). Here, we use a new approach based on the alternating direction method with multipliers (ADMM) [4] and numerical results show that in some situations, it is superior to the methods presented in [19, 27].

The rest of this paper is organised as follows. In Section 2, we establish equivalent formulations of the matrix nearness problem (1.1) and provide necessary and sufficient conditions for X^* to be a solution of (1.1). Iterative methods for solving the matrix nearness problem (1.1) are considered in Section 3 and their convergence is studied in Section 4.