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MULTIPLE VORTICES FOR THE SHALLOW WATER EQUATION IN TWO DIMENSIONS*†

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Dedicated to the 90th Birthday of Xiaqi Ding

Abstract

In this paper, we construct stationary classical solutions of the shallow water equation with vanishing Froude number Fr in the so-called lake model. To this end we need to study solutions to the following semilinear elliptic problem

$$\begin{cases} -\varepsilon^2 \text{div}\Big(\dfrac{\nabla u}{b}\Big) = b\Big(u - q\log\dfrac{1}{\varepsilon}\Big)_+^p, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

for small $\varepsilon > 0$, where $p > 1$, $\text{div}\big(\frac{\nabla q}{b}\big) = 0$ and $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain.

We show that if $\frac{q^2}{b}$ has m strictly local minimum (maximum) points $\bar{z}_i$, $i = 1, \cdots, m$, then there is a stationary classical solution approximating stationary m points vortex solution of shallow water equations with vorticity $\sum\limits_{i=1}^{m} \frac{2\pi q(\bar{z}_i)}{b(\bar{z}_i)}$. Moreover, strictly local minimum points of $\frac{q^2}{b}$ on the boundary can also give vortex solutions for the shallow water equation.

Keywords shallow water equation; free boundary; stream function; vortex solution

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1 Introduction and Main Results

In this paper, we consider fluid contained in a basin by a uniform gravitational acceleration g and fixed vertical lateral boundaries (that is, no sloping beaches).

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Suppose that (x, y) is horizontal spatial coordinate which is confined to a fixed bounded domain Ω with boundary $\partial\Omega$. The vertical coordinate is chosen so that the mean height of the fluid's free upper surface is at $z = 0$. Let $z = -b(x, y)$ give the fixed bottom topography, so b is a strict positive function over Ω. Let $z = h(x, y)$ be the free upper surface. We assume that both b and $\partial\Omega$ vary over distances L which are large compared to typical depth B, that is, the ratio $\delta = \frac{B}{L}$ is small.

Let $\mathbf{u}$ and w denote the horizontal and vertical components respectively of the fluid velocity. We will consider only those motion for which $\mathbf{u}$,w and h each vary in (x, y) over distances L, in other words, we will make the long-wave approximation. The "Froude number" is denoted as $Fr = \frac{U}{\sqrt{gB}}$, where U is the characteristic magnitude of $\mathbf{u}$. We will consider the case of small "Froude number" Fr and h is small compared to B. In such cases, from [1, 3, 4, 19], the leading-order evolution of $\mathbf{u}(x, y, t)$ and $h(x, y, t)$ will be governed by equations that have the non-dimensional form

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla h, \\ \operatorname{div}(b\mathbf{v}) = 0, \end{cases} \tag{1.1}$$

where ∇ is the horizontal gradient. Since these equations are applied to a domain which is shallow compared to its width and whose free surface exhibits negligible surface motion, they are called the "lake" equations (see [4], for instance).

The first equation in (1.1) can be rewritten in terms of the vorticity $\omega = \nabla \times \mathbf{v} := \frac{\partial \mathbf{v}_2}{\partial x} - \frac{\partial \mathbf{v}_1}{\partial y}$ as

$$\partial_t \mathbf{v} + \omega \times \mathbf{v} = -\nabla\Big(\frac{|\mathbf{v}|^2}{2} + h\Big),$$

$\omega \times \mathbf{v} = (-\mathbf{v}_2\omega, \mathbf{v}_1\omega)$. This model is analogous to the two-dimensional Euler equation for an idea incompressible fluid and has been recently studied by many authors. For instance, see [1, 3, 4, 19] and the references therein.

Recently, De Valeriola and Van Schaftingen [9] studied the desingularization of vortices for (1.1) with the stream function method, which consists in observing that if ψ satisfies

$$-\operatorname{div}\Big(\frac{\nabla\psi}{b}\Big) = bf(\psi),$$

for $f \in C^1(\mathbb{R})$, then $\mathbf{v} = \frac{\operatorname{curl}\psi}{b}$, and $h = -F(\psi) - \frac{|\mathbf{v}|^2}{2}$ with $F(s) = \int_0^s f(s)\mathrm{d}s$ form a stationary solution to the shallow water equation. Moreover, the velocity $\mathbf{v}$ is irrotational on the set where $f(\psi) = 0$. It is easy to see that if ψ_0 satisfies $-\operatorname{div}\big(\frac{\nabla\psi_0}{b}\big) = 0$, then $\mathbf{v}_0 = \frac{\operatorname{curl}\psi_0}{b}$, $h_0 = -\frac{|\mathbf{v}_0|^2}{2}$ is an irrotaional stationary solution to (1.1). In [9], they studied the asymptotics of solutions to

$$\begin{cases} -\varepsilon^2 \operatorname{div}\Big(\frac{\nabla\psi}{b}\Big) = b\psi_+^p, & \text{in } \Omega, \\ \psi = \psi_0 \ln\frac{1}{\varepsilon}, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$