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A High Order Central DG method of the Two-Layer Shallow Water Equations

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Abstract. In this paper, we focus on the numerical simulation of the two-layer shallow water equations over variable bottom topography. Although the existing numerical schemes for the single-layer shallow water equations can be extended to two-layer shallow water equations, it is not a trivial work due to the complexity of the equations. To achieve the well-balanced property of the numerical scheme easily, the two-layer shallow water equations are reformulated into a new form by introducing two auxiliary variables. Since the new equations are only conditionally hyperbolic and their eigenstructure cannot be easily obtained, we consider the utilization of the central discontinuous Galerkin method which is free of Riemann solvers. By choosing the values of the auxiliary variables suitably, we can prove that the scheme can exactly preserve the still-water solution, and thus it is a truly well-balanced scheme. To ensure the non-negativity of the water depth, a positivity-preserving limiter and a special approximation to the bottom topography are employed. The accuracy and validity of the numerical method will be illustrated through some numerical tests.

AMS subject classifications: 65M60, 65M12

Key words: Two-layer shallow water equations, Central DG method, Positivity-preserving and well-balanced, Still-water solution.

1 Introduction

The shallow water equations [19, 20] are widely adopted to model free-surface flows in rivers, channels, flood plains and coastal regions, however, the single-layer shallow water equations have the drawback of missing some physical dynamics in the vertical motion when studying stratified flow motions. Therefore, during the last decades,

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multi-layer shallow water equations have been attracted more attention and have become an important tool to study stratified flow motions, such as salinity-driven exchange flow motions [12], wave-mud interactions [17] and chaotic mixing of particles in layered flows [24].

In this paper, we focus on the numerical simulation of the two-layer shallow water equations, which are given by

$$\begin{cases} \dfrac{\partial h_1}{\partial t}+\dfrac{\partial (h_1u_1)}{\partial x}=0, \\ \dfrac{\partial (h_1u_1)}{\partial t}+\dfrac{\partial \left(h_1u_1^2+\frac{1}{2}gh_1^2\right)}{\partial x}=-gh_1\dfrac{\partial (h_2+b)}{\partial x}, \\ \dfrac{\partial h_2}{\partial t}+\dfrac{\partial (h_2u_2)}{\partial x}=0, \\ \dfrac{\partial (h_2u_2)}{\partial t}+\dfrac{\partial \left(h_2u_2^2+\frac{1}{2}gh_2^2\right)}{\partial x}=-gh_2\dfrac{\partial b}{\partial x}-gh_2\dfrac{\rho_1}{\rho_2}\dfrac{\partial h_1}{\partial x}, \end{cases} \tag{1.1}$$

in one-dimensional (1-D) space, and

$$\begin{cases} \dfrac{\partial h_1}{\partial t}+\dfrac{\partial (h_1u_1)}{\partial x}+\dfrac{\partial (h_1v_1)}{\partial y}=0, \\ \dfrac{\partial (h_1u_1)}{\partial t}+\dfrac{\partial \left(h_1u_1^2+\frac{1}{2}gh_1^2\right)}{\partial x}+\dfrac{\partial (h_1u_1v_1)}{\partial y}=-gh_1\dfrac{\partial (h_2+b)}{\partial x}, \\ \dfrac{\partial (h_1v_1)}{\partial t}+\dfrac{\partial (h_1u_1v_1)}{\partial x}+\dfrac{\partial \left(h_1v_1^2+\frac{1}{2}gh_1^2\right)}{\partial y}=-gh_1\dfrac{\partial (h_2+b)}{\partial y}, \\ \dfrac{\partial h_2}{\partial t}+\dfrac{\partial (h_2u_2)}{\partial x}+\dfrac{\partial (h_2v_2)}{\partial y}=0, \\ \dfrac{\partial (h_2u_2)}{\partial t}+\dfrac{\partial \left(h_2u_2^2+\frac{1}{2}gh_2^2\right)}{\partial x}+\dfrac{\partial (h_2u_2v_2)}{\partial y}=-gh_2\dfrac{\partial b}{\partial x}-gh_2\dfrac{\rho_1}{\rho_2}\dfrac{\partial h_1}{\partial x}, \\ \dfrac{\partial (h_2v_2)}{\partial t}+\dfrac{\partial (h_2u_2v_2)}{\partial x}+\dfrac{\partial \left(h_2v_2^2+\frac{1}{2}gh_2^2\right)}{\partial y}=-gh_2\dfrac{\partial b}{\partial y}-gh_2\dfrac{\rho_1}{\rho_2}\dfrac{\partial h_1}{\partial y}, \end{cases} \tag{1.2}$$

in two-dimensional (2-D) space. Herein, the subscripts 1 and 2 denote the upper and the lower layer in the system respectively (see Fig. 1). In these equations, ρ_i is a constant representing the density of the ith layer with $0<\rho_1<\rho_2$, h_i denotes the height of the ith layer, u_i,v_i are the vertically averaged horizontal velocity of the ith layer in the x- and y-directions, respectively, $i=1,2$, b is the bottom topography and g represents the gravitational constant.

Similar to single-layer shallow water equations, these equations admit still-water solutions

$$u_1=u_2=0, \quad h_1+h_2+b=\text{constant}, \quad h_2+b=\text{constant} \tag{1.3}$$

for 1-D case and

$$u_1=u_2=v_1=v_2=0, \quad h_1+h_2+b=\text{constant}, \quad h_2+b=\text{constant} \tag{1.4}$$