INTERNATIONAL JOURNAL OF NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 6, Pages 839–857

CONNECTION BETWEEN GRAD-DIV STABILIZED STOKES FINITE ELEMENTS AND DIVERGENCE-FREE STOKES FINITE ELEMENTS

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Abstract. In this paper, we use recently developed theories of divergence–free finite element schemes to analyze methods for the Stokes problem with grad-div stabilization. For example, we show that, if the polynomial degree is sufficiently large, the solutions of the Taylor–Hood finite element scheme converges to an optimal convergence exactly divergence–free solution as the grad-div parameter tends to infinity. In addition, we introduce and analyze a stable first-order scheme that does not exhibit locking phenomenon for large grad-div parameters.

Key words. Finite element methods, grad-div stabilization, divergence-free.

1. Introduction

Grad-div stabilization is a well-known and simple stabilization technique in numerical discretizations to improve mass conservation in simulations of incompressible flow. In its simplest form, the methodology adds the consistent term (written in strong form)

$$-\gamma\nabla(\nabla\cdot\boldsymbol{u})$$

to the momentum equations of the (Navier-)Stokes equations. Here, $\gamma > 0$ is a user-defined constant, which is referred to as the grad-div parameter. In addition to improving conservation of mass of the scheme, this stabilization technique may also improve the coupling errors of the velocity and pressure solutions. This can be advantageous for situations with large pressure gradients, e.g., in natural convection problems.

While enjoying many benefits, the use of grad-div stabilization comes with several practical disadvantages. These include a deterioration of the condition number and reduced sparsity of the algebraic system. Another disadvantage is the possible emergence of 'locking' for large grad-div parameters. Indeed, simply energy arguments show the discrete velocity solution satisfies $\|\nabla\cdot\boldsymbol{u}_h\| = O(\gamma^{-1})$, and therefore, in the limiting case, the discrete solution is divergence–free. If the discrete divergence–free subspace does not have rich enough approximation properties, then grad-div stabilization, while improving mass conservation, may lead to poor approximations.

The stability and convergence analysis for grad-div stabilization for incompressible flow have been explored in, e.g., [23, 9, 10, 27, 1]. These estimates, together with numerical simulations, provide a guide to choose optimal γ-values. For example, references [24, 21, 23, 4] suggests $\gamma = O(1)$ as the optimal value. On the other hand, numerical experiments in [12] and the analysis in [27, 1] suggest that the optimal choice may be much larger and depend on the finite element spaces, the mesh, and/or the viscosity of the model.

Received by the editors June 17, 2020 and, in revised form, September 11, 2020.
2000 *Mathematics Subject Classification.* 65N30, 65N15.

In another direction, and the path taken in this paper, is to identify and characterize the limiting solution as the grad-div parameter tends to infinity. For example, in [7, 19], it is shown that the Taylor–Hood finite element scheme on special (Clough-Tocher) triangulations, no locking occurs in the limiting case $\gamma \to \infty$, and the Taylor–Hood grad-div solution converges to the analogous (divergence–free) Scott–Vogelius solution.

The purpose of this paper is to extend and generalize the results in [7] by incorporating the recent theories of divergence–free finite element Stokes pairs. In this regard, we make two main contributions. First we show the absence of locking for the two-dimensional Taylor–Hood pair for a general class of meshes. In particular, we show that high–order Taylor–Hood pairs are generally locking-free. In addition, we show that the limiting (Taylor-Hood) solutions converge to the solution of the divergence–free Scott-Vogelius scheme, defined on general triangulations. The second contribution of the paper is the introduction and analysis of a new low–order and stable finite element pair that is locking–free. The velocity space is simply the linear Lagrange finite element space, and the pressure space consists of piecewise constants with respect to an auxiliary coarsened mesh.

The paper is organized as follows. In the next section, we introduce the notation and a framework for the grad-div finite element method for the Stokes problem. We show that the discrete solutions converge to a solution of a divergence–free method with rate $O(\gamma^{-1})$. In Section 3, we apply this framework to the two-dimensional Taylor–Hood elements. The general theme of the results is that additional mesh constraints are imposed for lower degree polynomial spaces. In Section 4, we define a stable first-order scheme for the Stokes problem, and show that the solutions converge to a divergence–free method as $\gamma \to \infty$. Finally, in Section 5 we provide some numerical experiments.

2. Notation and Framework

The Stokes equations defined on a polytope domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz continuous boundary $\partial\Omega$ is given by the system of equations

$$-\mu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{1b}$$

$$\boldsymbol{u} = 0 \qquad \text{on } \partial\Omega, \tag{1c}$$

where the $\boldsymbol{u}$ is the velocity, p the pressure, and ∇, Δ denote the gradient operator and vector Laplacian operators, respectively. In (1a), μ is the viscosity.

We define the following function spaces on Ω:

$$L^2(\Omega) := \{w : \Omega \mapsto \mathbb{R} : \ \|w\|_{L^2(\Omega)} := (\int_\Omega |w|^2\,dx)^{1/2} < \infty\},$$
$$H^m(\Omega) := \{w : \Omega \mapsto \mathbb{R} : \ \|w\|_{H^m(\Omega)} := (\sum_{|\beta|\le m} \|D^\beta w\|_{L^2(\Omega)}^2)^{1/2} < \infty\},$$

and set $(\cdot,\cdot)$ denote the inner product on $L^2(\Omega)$ and set $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. The analogous spaces with boundary conditions are given by

$$L_0^2(\Omega) := \{w \in L^2(\Omega) : \ \int_\Omega w\,dx = 0\},$$
$$H_0^m(\Omega) := \{w \in H^m(\Omega) : \ D^\beta w|_{\partial\Omega} = 0, \forall \beta : \ |\beta| \le m-1\}.$$

We denote the analogous vector-valued function spaces in boldface; for example $\boldsymbol{H}^1(\Omega) = H^1(\Omega)^d$ and $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^d$. We also define the space of $\boldsymbol{H}_0^1(\Omega)$