Journal of Computational Mathematics
Vol.39, No.2, 2021, 227–260.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1911-m2019-0044

QUADRATURE METHODS FOR HIGHLY OSCILLATORY SINGULAR INTEGRALS*

Jing Gao
School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China
Email: jgao@xjtu.edu.cn

Marissa Condon
School of Electronic Engineering, Dublin City University, Ireland
Email: marissa.condon@dcu.ie

Arieh Iserles
DAMTP, Centre for Mathematical Sciences, University of Cambridge, UK
Email: ai10@cam.ac.uk

Benjamin Gilvey and Jon Trevelyan[1)]
Department of Engineering, Durham University, UK
Email: benjamin.gilvey@durham.ac.uk, jon.trevelyan@durham.ac.uk

Abstract

We address the evaluation of highly oscillatory integrals, with power-law and logarithmic singularities. Such problems arise in numerical methods in engineering. Notably, the evaluation of oscillatory integrals dominates the run-time for wave-enriched boundary integral formulations for wave scattering, and many of these exhibit singularities. We show that the asymptotic behaviour of the integral depends on the integrand and its derivatives at the singular point of the integrand, the stationary points and the endpoints of the integral. A truncated asymptotic expansion achieves an error that decays faster for increasing frequency. Based on the asymptotic analysis, a Filon-type method is constructed to approximate the integral. Unlike an asymptotic expansion, the Filon method achieves high accuracy for both small and large frequency. Complex-valued quadrature involves interpolation at the zeros of polynomials orthogonal to a complex weight function. Numerical results indicate that the complex-valued Gaussian quadrature achieves the highest accuracy when the three methods are compared. However, while it achieves higher accuracy for the same number of function evaluations, it requires significant additional cost of computation of orthogonal polynomials and their zeros.

Mathematics subject classification: 65D32, 41A55.

Key words: Numerical quadrature, Singular highly oscillatory integrals, Asymptotic analysis, Boundary Element Method, Plane wave enrichment, Partition of Unity.

* Received February 19, 2019 / Revised version received October 21, 2019 / Accepted November 11, 2019 / Published online November 4, 2020 /

1) Corresponding author

1. Introduction

In this paper, we shall consider four kinds of highly oscillatory singular integrals: with singular non-oscillatory component,

$$I[f] = \int_0^b x^{-\alpha} f(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x, \tag{1.1}$$

$$I[f] = \int_0^b \log x f(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x, \tag{1.2}$$

and where the singularity appears in the phase function [22]

$$I[f] = p \int_0^b x^{-\alpha} f(x) \mathrm{e}^{\mathrm{i}\omega x^{-p}} \mathrm{d}x, \quad \alpha \in (0,1), \quad p > 0, \tag{1.3}$$

$$I[f] = \int_0^b \log x f(x) \mathrm{e}^{\mathrm{i}\omega \log x} \mathrm{d}x, \tag{1.4}$$

where f and g are sufficiently smooth functions, independent of $|\omega| \gg 1$.

Oscillatory integrals are encountered in a wide range of applications in science and engineering. In cases where the integrand exhibits a singularity of one of the above types, this further complicates the evaluation of the integral. Such integrals arise in applications in electromagnetics [17, 18] and electromagnetic shielding problems [10]. In the present work we focus on the frequency-domain, wave-enriched boundary integral formulations as a particular example to motivate this work.

Acoustic wave propagation problems in 2-D unbounded regions are governed by the Helmholtz differential equation

$$\Delta u(\mathrm{p}) + \omega^2 u(\mathrm{p}) = 0, \quad \mathrm{p} \in \mathbb{R}^2 \backslash \overline{\Omega},$$

where $u \in \mathbb{C}$ is the acoustic potential sought as the solution, p is the position, λ is the wave length, $\omega = 2\pi/\lambda$, Ω is a bounded obstacle with a boundary Γ and $\overline{\Omega} = \Omega \cup \Gamma$. Three classical numerical methods that are widely used in industry for the analysis of wave propagation problems are the Finite Difference Method (FDM), Finite Element Method (FEM) and Boundary Element Method (BEM). However, complications arise for wave scattering problems in infinite domains. In particular, FDM and FEM discretisations require truncation of the domain and the application of some non-reflecting boundary conditions on the artificial exterior boundary. BEM formulations are therefore often preferable to volumetric discretisations for these problems since the meshing is restricted to the boundary of the scatterer(s), leading to the efficient solution of problems in unbounded geometries.

The first step of the BEM is to transform the Helmholtz equation into a boundary integral equation. The Robin boundary condition

$$\nabla u(\mathrm{q}) \cdot n(\mathrm{q}) = \gamma u(\mathrm{q}) + \beta, \quad \mathrm{q} \in \Gamma$$

is imposed, where γ and β may be derived from the impedance characteristics of the material(s) forming the scatterer boundaries, and $n(\mathrm{q})$ is the unit normal at q pointing outward from the solution domain. Applying Green' second theorem and Sommerfeld radiation conditions yields the boundary integral equation

$$\frac{u(\mathrm{p})}{2} + \int_\Gamma \left[\frac{\partial G(\mathrm{p},\mathrm{q})}{\partial \mathrm{n}(\mathrm{q})} - \gamma G(\mathrm{p},\mathrm{q}) \right] u(\mathrm{q}) \mathrm{d}\Gamma_{\mathrm{q}} = \int_\Gamma \beta G(\mathrm{p},\mathrm{q}) \mathrm{d}\Gamma(q) + u^i(\mathrm{p}),$$