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Regularity and Rigidity for Nonlocal Curvatures in Conformal Geometry

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Dedicated to Professors Sun-Yung Alice Chang and Paul C.-P. Yang on their 70th birthdays

Abstract. In this paper, we will explore the geometric effects of conformally covariant operators and the induced nonlinear curvature equations in certain nonlocal nature. Mainly, we will prove some regularity and rigidity results for the distributional solutions to those equations.

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1 Introduction

In conformal geometry, a primary goal is to understand both analytic and geometric properties of the conformally covariant operators. Specifically, let (M^n,g_0) be a Riemannian manifold of dimension n and equipped with a conformal structure $[g_0]$. For example, in the case of second order differential operators, the most classical conformally covariant operator is the conformal Laplacian. For $n\geq 3$, it is defined as

$$P_{g_0}=L_{g_0}=-\Delta_{g_0}+\frac{(n-2)}{4(n-1)}R_{g_0}, \tag{1.1}$$

where $R_{g_0}=(\frac{n-2}{4(n-1)})^{-1}L_{g_0}(1)$ is the scalar curvature of the metric g_0. In the context of *conformally compact Einstein manifolds*, geometric scattering theory gives a much more general manner to study the conformally covariant operators. That is, for each $\gamma\in(0,n)$,

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there is a well-defined conformally covariant operator $P_{2\gamma}$ of order 2γ which is called the fractional GJMS operator, see Definition 1.2 below. Similarly, the scalar function $Q_{2\gamma} \equiv (\frac{n-2\gamma}{2})^{-1} P_{2\gamma}(1)$ is called the nonlocal Q curvature of order 2γ. In particular, when $\gamma = 2$, the operator P_4 is called the Paneitz operator. Correspondingly, Q_4 is called Branson's Q curvature which naturally arises from the Chern-Gauss-Bonnet integral on a 4-manifold and hence deeply related to the geometry and topology of the underlying manifold. There are a lot of fundamental works in this direction (see, e.g., [1, 3, 6, 18–21, 24] and the references therein).

To start with, we introduce some background materials of the conformally compact Einstein manifolds and the precise definition of the nonlocal curvature $Q_{2\gamma}$.

Definition 1.1. Given a pair of smooth manifolds (X^{n+1}, M^n) with $M^n \equiv \partial X^{n+1}$, we say a complete Einstein metric g_+ on X^{n+1} is *conformally compact Einstein* if it satisfies

$$\mathrm{Ric}_{g_+} \equiv -n g_+ \tag{1.2}$$

and there is a conformal metric $\bar{g} = u^2 g_+$ which smoothly extends to the boundary M^n with a restriction

$$h_0 \equiv u^2 g_+|_{M^n}. \tag{1.3}$$

The conformal manifold $(M^n, [h_0])$ is called the *conformal infinity*.

Establishing effective connections between the conformal structure of the conformal infinity $(M^n, [h_0])$ and the Riemannian structure of the Einstein filling-in (X^{n+1}, g_+) is always a central topic of conformal geometry and the theory of AdS/CFT correspondence. A crucial tool in understanding the above structure is the conformally covariant operators defined on a conformally compact Einstein manifold. Due to Graham-Zworski [16] and Chang-González [5], there are a family of conformally covariant operators called the *fractional GJMS operators* (see [2, 5]). Specifically, in the context of Definition 1.1, we define the operator as below.

Definition 1.2 (Fractional GJMS operator)**.** Given any real number $\gamma \in (0, \frac{n}{2})$,

$$P_{2\gamma}[g_+, h_0] \equiv 2^{2\gamma} \cdot \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} \cdot S(\frac{n}{2} + \gamma), \tag{1.4}$$

where S is the scattering operator which is essentially a Dirichlet-to-Neumann operator (we refer the reader to [5] and [16] for the more detailed definition of the scattering operator).

In this case, let $\hat{h} = v^{\frac{4}{n-2\gamma}} h_0$ and let $\widehat{P}_{2\gamma}$ be the fractional GJMS operator with respect to $\hat{h}$, then

$$\widehat{P}_{2\gamma}(u) = v^{-\frac{n+2\gamma}{n-2\gamma}} P_{2\gamma}(uv), \tag{1.5}$$