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OSCILLATION OF THIRD-ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS*†

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Abstract

In this paper, we give some new criteria for the asymptotic behavior and oscillation of third-order delay differential equation. The oscillation of the studied equation is studied under two conditions, and our results improve some ones in Džurina et al. (2018). Some examples are given to illustrate the main results with Euler-type differential equations.

Keywords nonlinear differential equation; delay; third-order; oscillation
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1 Introduction

In 2012, Elabbasy ([3]) presented oscillation criteria for third order delay nonlinear differential equation of the form

$$\left[a_2(t)\left\{(a_1(t)(x'(t))^{\alpha_1})'\right\}^{\alpha_2}\right]' + q(t)f(x(g(t))) = 0.$$

In 2018, Džurina ([1]) gave new oscillation criteria for third-order delay differential equations with noncanonical operators of the form

$$\left(r_2(r_1y')'\right)'(t) + q(t)y(\tau(t)) = 0, \quad t \geq t_0 > 0.$$

In this paper, we consider the oscillatory behavior of solutions to a third-order nonlinear delay differential equation of the form

$$\left(r_2(t)((r_1(t)(y'(t))^{\alpha_1})')^{\alpha_2}\right)' + q(t)y(\tau(t)) = 0, \quad t \geq t_0 > 0. \tag{1.1}$$

Throughout the whole paper, we assume that

(H_1) α_1,α_2 are quotient of odd positive integers;

(H_2) the functions $r_1, r_2 \in c([t_0,\infty),(0,\infty))$ satisfy

$$\pi_1(t_0) := \int_{t_0}^{\infty} r_1^{-\frac{1}{\alpha_1}}(s)\mathrm{d}s < \infty, \qquad \pi_2(t_0) := \int_{t_0}^{\infty} r_2^{-\frac{1}{\alpha_2}}(s)\mathrm{d}s < \infty;$$

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(H$_3$) $q \in c([t_0,\infty),[0,\infty))$ does not vanish eventually;

(H$_4$) $\tau \in c^1([t_0,\infty),(0,\infty))$, $\tau(t) \le t$, $\tau'(t) > 0$, and $\lim\limits_{t\to\infty} \tau(t) = \infty$.

By a solution of equation (1.1), we mean a function $y(t) : [T_y,\infty) \to \mathbb{R}$, $T_y \ge t_0$ such that $y(t)$, $r_1(t)(y'(t))^{\alpha_1}$, $r_2(t)\big((r_1(t)(y'(t))^{\alpha_1})'\big)^{\alpha_2}$ are continuous and differentiable for all $t \in [T_y,\infty)$, and satisfies (1.1) on $[T_y,\infty)$. We only need to consider those solutions of (1.1) which exist on some half-line $[T_y,\infty)$ and satisfy the condition

$$\sup\{|y(t)| : T \le t < \infty\} > 0$$

for any $T \ge T_y$. In the sequel, we assume that (1.1) possesses such a solution.

As is customary, a solution $y(t)$ of (1.1) is called oscillatory if it has arbitrary large zeros on $[T_y,\infty)$. Otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions oscillate.

Following classical results of Kiguradze and Kondrat'ev [6], we say that (1.1) has property A if any solution y of (1.1) is either oscillatory or satisfies $\lim\limits_{t\to\infty} y(t) = 0$. Instead of calling property A, some authors say that equation is almost oscillatory.

For the sake of brevity, we define the operators

$$L_0y = y,\quad L_1y = r_1(y')^{\alpha_1},\quad L_2y = r_2\big((r_1(y')^{\alpha_1})'\big)^{\alpha_2},\quad L_3y = \big(r_2((r_1(y')^{\alpha_1})')^{\alpha_2}\big)'.$$

2 Main Results

As usual, all functional inequalities considered in this paper are supposed to hold eventually, that is, they are satisfied for all t large enough.

Without loss of generality, we need only to consider eventually positive solutions of (1.1).

The following lemma on the structure of possible nonoscillatory solutions of (1.1) plays a crucial role in the proofs of the main results.

Lemma 2.1 *Assume* (H$_1$)-(H$_4$) *and that y is an eventually positive solution of* (1.1). *Then there exists a $t_1 \in [t_0,\infty)$ such that y is one of the following cases:*

$$\begin{aligned}
&\text{case (1)}: \quad y>0,\quad L_1y<0,\quad L_2y<0,\quad L_3y<0;\\
&\text{case (2)}: \quad y>0,\quad L_1y<0,\quad L_2y>0,\quad L_3y<0;\\
&\text{case (3)}: \quad y>0,\quad L_1y>0,\quad L_2y>0,\quad L_3y<0;\\
&\text{case (4)}: \quad y>0,\quad L_1y>0,\quad L_2y<0,\quad L_3y<0,
\end{aligned}$$

for $t \ge t_1$.

The proof is straightforward and hence is omitted.

Theorem 2.1 *Assume* (H$_1$)-(H$_4$). *If*

$$\int_{t_0}^{\infty} r_1^{-\frac{1}{\alpha_1}}(v)\left(\int_{t_0}^{v} r_2^{-\frac{1}{\alpha_2}}(u)\left(\int_{t_0}^{u} q(s)\mathrm{d}s\right)^{\frac{1}{\alpha_2}}\mathrm{d}u\right)^{\frac{1}{\alpha_1}}\mathrm{d}v = \infty, \tag{2.1}$$

then (1.1) *has property A.*