Ann. of Appl. Math.
36:3(2020), 309-330

NEW OSCILLATION CRITERIA FOR THIRD-ORDER HALF-LINEAR ADVANCED DIFFERENTIAL EQUATIONS*†

Jianli Yao‡, Xiaoping Zhang, Jiangbo Yu
(*School of Science, Shandong Jianzhu University, Ji'nan 250101, Shandong, PR China*)

Abstract

The theme of this article is to provide some sufficient conditions for the asymptotic property and oscillation of all solutions of third-order half-linear differential equations with advanced argument of the form

$$\left(r_2(t)((r_1(t)(y'(t))^{\alpha})')^{\beta}\right)' + q(t)y^{\gamma}(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

where $\int^{\infty} r_1^{-\frac{1}{\alpha}}(s)\mathrm{d}s < \infty$ and $\int^{\infty} r_2^{-\frac{1}{\beta}}(s)\mathrm{d}s < \infty$. The criteria in this paper improve and complement some existing ones. The results are illustrated by two Euler-type differential equations.

Keywords third-order differential equation; advanced argument; oscillation; asymptotic behavior; noncanonical operators

2000 Mathematics Subject Classification 34C10; 34K11

1 Introduction

In 2019, Chatzarakis ([1]) offered sufficient conditions for the oscillation and asymptotic behavior of second-order half-linear differential equations with advanced argument of the form

$$\left(r(y')^{\alpha}\right)'(t) + q(t)y^{\alpha}(\sigma(t)) = 0,$$

where $\int^{\infty} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s < \infty$.

In 2018, Džurina ([2]) presented new oscillation criteria for third-order delay differential equations with noncanonical operators of the form

$$\left(r_2\left(r_1y'\right)'\right)'(t) + q(t)y(\tau(t)) = 0, \quad t \geq t_0 > 0.$$

*This work was supported by Youth Program of National Natural Science Foundation of China under Grant 61304008 and Youth Program of Natural Science Foundation of Shandong Province under Grant ZR2013FQ033.

†Manuscript received March 27, 2020; Revised August 5, 2020

‡Corresponding author. E-mail: yaojianli@sdjzu.edu.cn

In this paper, we consider the oscillatory and asymptotic behavior of solutions to the third-order half-linear advanced differential equations of the form

$$\left(r_2(t)((r_1(t)(y'(t))^{\alpha})')^{\beta}\right)' + q(t)y^{\gamma}\left(\sigma(t)\right) = 0, \quad t \geq t_0 > 0. \tag{1.1}$$

Throughout the whole paper, we assume that

(H_1) α, β and γ are quotients of odd positive integers;

(H_2) the functions $r_1, r_2 \in C\left([t_0,\infty),(0,\infty)\right)$ are of noncanonical type (see Trench [2]), that is,

$$\pi_1\left(t_0\right) := \int_{t_0}^{\infty} r_1^{-\frac{1}{\alpha}}(s)\mathrm{d}s < \infty, \quad \pi_2\left(t_0\right) := \int_{t_0}^{\infty} r_2^{-\frac{1}{\beta}}(s)\mathrm{d}s < \infty;$$

(H_3) $q \in C\left([t_0,\infty),[0,\infty)\right)$ does not vanish eventually;

(H_4) $\sigma \in C^1\left([t_0,\infty),(0,\infty)\right)$, $\sigma(t) \geq t$, $\sigma'(t) \geq 0$ for all $t \geq t_0$.

By a solution of equation (1.1), we mean a nontrivial real valued function $y \in C([T_x,\infty),\mathbb{R})$, $T_x \geq t_0$, which has the property that y, $r_1(y')^{\alpha}$, $r_2\big((r_1(y')^{\alpha})'\big)^{\beta}$ are continuous and differentiable for all $t \in [T_x,\infty)$, and satisfy (1.1) on $[T_x,\infty)$. We only need to consider those solutions of (1.1) which exist on some half-line $[T_x,\infty)$ and satisfy the condition

$$\sup\{|y(t)| : T \leq t < \infty\} > 0$$

for any $T \geq T_x$. In the sequel, we assume that (1.1) possesses such solutions.

As is customary, a solution $y(t)$ of (1.1) is called oscillatory if it has arbitrary large zeros on $[T_x,\infty)$. Otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions oscillate.

Following classical results of Kiguradze and Kondrat'ev [3], we say that (1.1) has property A if any solution y of (1.1) is either oscillatory or satisfies $\lim\limits_{t\to\infty} y(t) = 0$, which is also called that equation (1.1) is almost oscillatory.

For brevity, we define operators

$$L_0y = y, \quad L_1y = r_1(y')^{\alpha}, \quad L_2y = r_2\big((r_1(y')^{\alpha})'\big)^{\beta}, \quad L_3y = \big(r_2((r_1(y')^{\alpha})')^{\beta}\big)'.$$

Also, we use the symbols $\uparrow$ and $\downarrow$ to indicate whether the function is nondecreasing and nonincreasing, respectively.

2 Main Results

As usual, all functional inequalities considered in this paper are supposed to hold eventually, that is, they are satisfied for all t large enough.

Without loss of generality, we need only to consider eventually positive solutions of (1.1), since if y satisfies (1.1), so does $-y$.