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Nonparametric Mean Curvature Flow with Nearly Vertical Contact Angle Condition

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Abstract. For any bounded strictly convex domain Ω in $\mathbb{R}^n$ with smooth boundary, we find the prescribed contact angle which is nearly perpendicular such that nonparametric mean curvature flow with contact angle boundary condition converge to ones which move by translation. Subsequently, the existence and uniqueness of smooth solutions to the capillary problem without gravity on strictly convex domain are also discussed.

AMS subject classifications: 35K59, 35J93.

Key words: Mean curvature flow, prescribed contact angle, asymptotic behavior, capillary problem.

1 Introduction

In this paper, we are interested in the study of the evolution of graphs defined over bounded strict convex domains $\Omega\subset\mathbb{R}^n$ by the nonparametric mean curvature flow, whose speed in the direction of their normal is equal to their mean curvature and with a prescribed contact angle to $\partial\Omega$.

Various results have been obtained for mean curvature flow of hypersurfaces with Dirichlet boundary conditions [26], zero-Neumann boundary condition [15], [18] and

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general Neumann boundary condition [25]. We study the evolution of graphs for $u=u(x,t)$ with the speed depending on the mean curvature of the surface $\{(x,u(x,t)):x\in\Omega\}$ and with the prescribed contact angle boundary condition, that is,

$$\begin{cases} u_t=\sqrt{1+|Du|^2}\mathcal{H}(u) & \text{in } \Omega\times(0,\infty), \\ \langle\gamma,\nu\rangle=\cos\theta & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x) & \text{on } \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$, $n\geq 2$, is a compact domain with smooth boundary $\partial\Omega$, $\theta:\partial\Omega\to\mathbb{R}$ is the angle (*contact angle*) between the graph and the boundary, given by $\langle\gamma,\nu\rangle=\cos\theta$, which is equivalent to $u_\nu=-\cos\theta\sqrt{1+|Du|^2}$, where ν is the unit inner normal of $\partial\Omega$. Remark that one may extend θ to $\overline{\Omega}$ with $\theta\in C^\infty(\overline{\Omega})$. And $u_0(x)$ is also a smooth function satisfying the compatible condition

$$u_{0,\nu}=-\cos\theta\sqrt{1+|Du_0|^2} \quad \text{on} \quad \partial\Omega.$$

While $\mathcal{H}$ is the mean curvature operator

$$\mathcal{H}(u):=\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right),$$

and γ is the upward normal of the graph $\{(x,u(x,t)):x\in\Omega\}$, which is given by

$$\gamma:=\frac{(-Du,1)}{\sqrt{1+|Du|^2}},$$

and we denote by $Q_T:=\Omega\times[0,T)$ for convenience.

For the prescribed contact angle boundary condition, a more general type of problem is to study the following equation, which has an extra term $F(x,u,Du)$ (some called the transport term) compared to (1.1), i.e.

$$u_t=\sqrt{1+|Du|^2}\mathcal{H}(u)-F(x,u,Du) \quad \text{in} \quad Q_T. \tag{1.2}$$

Guan [14] proved the global existence of solutions to (1.2) with prescribed contact angle condition for general bounded domain Ω. Recently, Zhou generalized Guan's results to the domain Ω on Riemannian manifold in [30].

As for studying the asymptotic behavior of $u(x,t)$ in (1.2), Guan [14] or Zhou [30] only obtained the convergence results for $F(x,u,Du)$ with specific form, say $F:=\phi(x,u)\cdot\sqrt{1+|Du|^2}$ with $\phi_u\geq c_0>0$, which excluded $F\equiv 0$. In [15], Huisken studied the fixed vertical contact angle case of (1.1), i.e. $\theta(x)\equiv\frac{\pi}{2}$, so $u_\nu=0$ on $\partial\Omega$. By using the Sobolev-type inequalities and an iteration method, Huisken proved that the solution $u(\cdot,t)$ of (1.1) converges to a constant function as $t\to+\infty$. For the non-perpendicular case, Altschuler