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Existence Results for Super-Liouville Equations on the Sphere via Bifurcation Theory

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Abstract. We are concerned with super-Liouville equations on S^2, which have variational structure with a strongly-indefinite functional. We prove the existence of nontrivial solutions by combining the use of Nehari manifolds, balancing conditions and bifurcation theory.

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Key words: Super-Liouville equations, existence results, bifurcation theory, critical groups.

1 Introduction

In this paper we study the super-Liouville equations, arising from Liouville field theory in supergravity. Recall that the classical Liouville field theory describes the matter-induced gravity in dimension two: the super-Liouville field theory is a supersymmetric generalization of the classical one, by taking the spinorial super-partner into account, so that the bosonic and fermionic fields couple under the supersymmetry principle. Such models also play a role in superstring theory. For the physics of the Liouville field theory and super-Liouville field theory as well as their relations, one can refer to [9,39–42], and for the applications of Liouville field theory in other models of mathematical physics [44, 46–48] and the references therein. It is almost impossible to have a complete references

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for the related theory. However, the existence theory of regular solutions of the super-Liouville equations on closed Riemann surfaces, especially on the sphere, is still far from satisfactory.

Liouville equations also have a relevant role in two-dimensional geometry. For example, on a Riemannian surface (M^2,g), the Gaussian curvature K of a conformal metric $\widetilde{g}:=e^{2u}g$, with $u\in C^\infty(M)$, is given by

$$K_{\widetilde{g}}=e^{-2u}(K_g-\Delta_g u). \tag{1.1}$$

Conversely, we have the *prescribed curvature problem*: which functions $\widetilde{K}$ can be the Gaussian curvatures of a Riemannian metric conformal to g? If M is a closed surface, the problem reduces to solving equation (1.1) in u for $K_{\tilde{g}}=\widetilde{K}$ assigned. This question has been widely studied in the last century, and the solvability of (1.1) depends on the geometry and the topology of the surface. For a surface with nonzero genus, this can be solved variationally, as long as $\widetilde{K}$ satisfies some mild constraints, see [6,32,43]. However when the genus is zero, namely M is a topological two-sphere, the problem has additional difficulties arising from the non-compactness of the automorphism group. Actually, since there is only one conformal structure on $\mathbb{S}^2$, we can take without loss of generality the standard round metric $g=g_0$, which is the one induced by the embedding $\mathbb{S}^2\subset\mathbb{R}^3$ with Gaussian curvature $K_{g_0}=1$. Let $x=(x^1,x^2,x^3)$ be the standard coordinates of $\mathbb{R}^3$. It was shown in [32] that a necessary condition for $\widetilde{K}$ to admit a solution u of (1.1) is that

$$\int_{\mathbb{S}^2}\langle\nabla\widetilde{K},\nabla x^j\rangle e^{2u}\,\mathrm{d}vol=0, \qquad \forall\, 1\le j\le 3,$$

where the volume form dvol and the gradient are taken with respect to g_0. The above formula shows that, for example, affine functions cannot be prescribed conformally as Gaussian curvatures.

One of the first existence results for the problem on the sphere is due to Moser, see [38]: he proved that there exist solutions provided that $\widetilde{K}$ is an antipodally-symmetric function. Other important results were proven in [12,13], removing the symmetry condition and replacing it with an index-counting condition or some assumption of min-max type, see also [14]. One fundamental tool in proving such results was an improved Moser-Trudinger inequality derived in [5] for functions satisfying a *balancing condition*, namely for which the conformal volume has zero center of mass in $\mathbb{R}^3$ (where $\mathbb{S}^2$ is embedded). This fact allowed to show that whenever solutions (or approximate solutions) of (1.1) blow-up, they develop a single-bubbling behavior. With this information at hand, existence results were derived via asymptotic estimates and Morse-theoretical results. We should also mention that there are natural generalizations to higher dimensions, see e.g. [36] and references therein.

Recently Jost et al. in [26] considered a mathematical version of the super-Liouville equations on surfaces. Given a Riemann surface M with metric g, and $S\to M$ the spinor