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Spectral Properties of an Energy-Dependent Hamiltonian

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Abstract. We study spectral properties of a quantum Hamiltonian with a complex-valued energy-dependent potential related to a model introduced in physics of nuclear reactions [30] and we prove that the principle of limiting absorption holds at any point of a large subset of the essential spectrum. When an additional dissipative or smallness hypothesis is assumed on the potential, we show that the principle of limiting absorption holds at any point of the essential spectrum.

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1 Introduction

Our goal in this paper is to study spectral properties of the energy-dependent operator

$$H(\rho) \equiv -\Delta + U_0 + U(\rho) \tag{1.1}$$

in $\mathbb{R}^3$, where Δ is the Laplacian in $\mathbb{R}^3$, $\rho \in \mathbb{C}$ is the spectral parameter (the associated energy is $\zeta = \rho^2$), U_0 denotes multiplication operator by the function

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$U_0(x):\mathbb{R}^3\to\mathbb{R}$ and $U(\rho):=V(\rho)-iW(\rho)$ denotes multiplication operator by the function $U(x,\rho):\mathbb{R}^3\times\mathbb{C}\to\mathbb{C}$ with $U(x,\rho):=V(x,\rho)-iW(x,\rho)$.

We also suppose that V and W are factorized as follows

$$V(x,\rho)=V_R(x)V_R'(\rho),\quad W(x,\rho)=W_I(x)W_I'(\rho). \tag{1.2}$$

Such an operator appears in physics of nuclear reactions [7]: U_0 is a (possibly long-range) external potential (Coulomb) and $U(\rho)$, the energy-dependent part (possibly long-range), is called Optical Potential in this context (see [2] for a physical introduction). In [3] spectral properties of a related 1D model relying on specific space-energy hypotheses for V and W (specified in [30]) have been studied and our aim in the present paper is to generalize this study to the physical dimension $d=3$ when dissipation may be possibly taken into account.

It is worth to observe, using inverse Laplace transform that the operator $H(\rho)$ can be derived from a Schrödinger equation perturbed by a non-local term involving a convolution with respect to time leading to the problem

$$\begin{aligned} i\partial_t\phi(x,t)=&-\Delta\phi(x,t)+u_0(x)\phi(x,t)\\ &+\int_0^\infty\big[v(x,t-s)-iw(x,t-s)\big]\phi(x,s)ds, \end{aligned} \tag{1.3}$$

$$\phi(x,t)|_{t=0}=\phi_0(x) \tag{1.4}$$

for $(x,t)\in\mathbb{R}^3\times\mathbb{R}_+$ and for real valued u_0, v and w, where v and w satisfy a causality assumption

$$\text{“For any } x\in\mathbb{R}^3\ \ v(x,t)\ \text{ and }\ w(x,t)\ \text{ vanish when } t<0.\text{”} \tag{1.5}$$

It will be the purpose of a future work [4] to study solutions of (1.3)-(1.4) and their large time asymptotics. Just mention that this problem has yet be studied by Mochizuki and Motai [29] in the local case $v(x,s)-iw(x,s)\equiv(\tilde{v}(x)-i\tilde{w}(x))\delta(t-s)$.

We make the following hypotheses on functions $U_0(\cdot)\in C^\infty(\mathbb{R}^3,\mathbb{R})$, $V(\cdot,\rho)\in C^\infty(\mathbb{R}^3,\mathbb{R})$ and $W(\cdot,\rho)\in C^\infty(\mathbb{R}^3,\mathbb{R})$, which comply with the expressions given in [30].

Condition 1. There exist constants $C_\alpha>0$, $a_0>0$, $a_1>0$, $a_2>0$ and $R_0>0$ such that uniformly in the positive half-plane $\mathbb{C}_+\equiv\{\Im m(\rho)>0\}$ and for any $\alpha\in\mathbb{N}^3$ and for $|x|\geq R_0$

$$\big|\partial_x^\alpha U_0(x)\big|\leq C_\alpha e^{-a_0|x|}, \tag{1.6}$$

$$\big|\partial_x^\alpha V_R(x)\big|\leq C_\alpha e^{-a_1|x|}, \tag{1.7}$$

$$\big|\partial_x^\alpha W_I(x)\big|\leq C_\alpha e^{-a_2|x|}. \tag{1.8}$$