Commun. Math. Res.
doi: 10.4208/cmr.2020-0043

Vol. **37**, No. 2, pp. 209-235
May 2021

On the J.L. Lions Lemma and its Applications to the Maxwell-Stokes Type Problem and the Korn Inequality

Junichi Aramaki*

Division of Science, Tokyo Denki University, Hatoyama-machi, Saitama, 350-0394, Japan.

Received 21 November 2020; Accepted 11 January 2021

Abstract. In this paper, we consider the equivalent conditions with L^p-version $(1<p<\infty)$ of the J.L. Lions lemma. As applications, we first derive the existence of a weak solution to the Maxwell-Stokes type problem and then we consider the Korn inequality. Furthermore, we consider the relation to other fundamental results.

AMS subject classifications: 35A01, 35D30, 35J62, 35Q61, 35A15

Key words: J.L. Lions lemma, de Rham theorem, Maxwell-Stokes type problem, multiply-connected domain.

1 Introduction

Assume that Ω is a domain of $\mathbb{R}^d$. In this paper, this means that Ω is a bounded and connected open subset of $\mathbb{R}^d$ whose boundary $\Gamma=\partial\Omega$ is Lipschitz-continuous and Ω is locally on the same side of Γ. The classical J.L. Lions lemma asserts that any distribution in the space of $H^{-1}(\Omega)$ with the gradient (in the distribution sense) belonging to $\boldsymbol{H}^{-1}(\Omega)$ is a function in $L^2(\Omega)$.

Amrouche *et al.* [1] derived the equivalent conditions with the J.L. Lions lemma. The conditions are the classical and the general J.L. Lions lemma, the Nečas

*Corresponding author. *Email address:* `aramaki@hctv.ne.jp` (J. Aramaki)

inequality, the coarse version of the de Rham theorem, the surjectivity of the operator $\operatorname{div}: \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$ and an approximation lemma. Some of these equivalent properties can be given as a "direct" proof.

However, these equivalent conditions of L^2-version of the J.L. Lions lemma are insufficient for considering the Maxwell-Stokes type system containing p-curlcurl operator. Thus it is important for us to improve the result to the L^p-version of the equivalent relations with the J.L. Lions lemma.

One of our purpose of this paper is to show the existence of a weak solution to the Maxwell-Stokes type equation. The existence depends deeply on the nonlinearity of the equation and the shape of the domain. We allow the domain Ω to be a multiply-connected domain with holes. To show the existence, we apply the equivalent conditions with the J.L. Lions lemma, in particular, the coarse version of the de Rham theorem. For example, L^p-version of the Nečas inequality can be found in Nečas [19, Theorem 1], Geymonat and Suquet [14, Lemma 1] and Amrouche and Girault [2]. For the Maxwell type system containing p-curlcurl operator, see Miranda *et al.* [17, 18] and Aramaki [6], and for the Maxwell-Stokes type system, see Pan [20] and Aramaki [9] and references therein.

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we derive L^p-version of the J.L. Lions lemma and its equivalent relations. In Section 4, we give an application on the existence of a weak solution to the Maxwell-Stokes type problem. Section 5 is devoted to consider the Korn inequality. In the Appendix A, we give a relation between the J.L. Lions lemma and the de Rham theorem, and in the Appendix B, we give a relation between the J.L. Lions lemma and a weak version of Poincaré lemma.

2 Preliminaries

In this section, we shall state some preliminaries that are necessary in this paper. Let Ω be a domain in $\mathbb{R}^d$ $(d \geq 2)$ (which means a bounded, connected open subset of $\mathbb{R}^d$ with a Lipschitz-continuous boundary Γ), let $1 < p < \infty$ and let p' be the conjugate exponent i.e., $(1/p)+(1/p')=1$. From now on we use $\mathcal{D}(\Omega)$, $L^p(\Omega)$, $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$, $W^{-m,p}(\Omega) = W_0^{m,p}(\Omega)'$ (the dual space of $W_0^{m,p}(\Omega)$, $m \geq 0$, integer), $W^{s,p}(\Gamma)$ $(s \in \mathbb{R})$, and so on, for the standard real C^∞ functions with compact supports in Ω, L^p and Sobolev spaces of real valued functions. For any above space B, we denote B^d by boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vector and vector-valued functions. Occasionally, we also use the same character for matrix values functions, and we denote the standard inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^d$ by $\boldsymbol{a} \cdot \boldsymbol{b}$. We denote the space of distributions in