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The Center Conditions and Hopf Cyclicity for a 3D Lotka-Volterra System*

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Abstract The main objective of this paper is not only to find necessary and sufficient conditions for the existence of a center on a local center manifold for a three dimensional Lotka-Volterra system, but also to determine the maximum number of limit cycles that can bifurcate from the positive equilibrium as a fine focus. Firstly, the singular point quantities are computed and simplified to obtain necessary conditions for local integrability, and Darboux method is applied to show the sufficiency. Then, the Hopf bifurcation on the center manifold is investigated, from this, the conclusion of at most five small limit cycles generated in the vicinity of the equilibrium is obtained. To the best of our knowledge, this is the first case with five possible limit cycles for the cyclicity of 3D Lotka-Volterra systems.

Keywords 3D Lotka-Volterra system, Hopf bifurcation, Center problem, Singular point quantities.

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1. Introduction

We consider the three-dimensional (3D) Lotka-Volterra system:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i(r_i - \sum_{j=1}^{3} a_{ij}x_i), \quad i = 1, 2, 3 \tag{LV}$$

when $r_i > 0, a_{ij} > 0$ $(i, j = 1, 2, 3)$, (LV) is called the competitive 3D LV system, which is very classical one to describe the relations of three species that share and compete for the same resources, habitat or territory (interference competition). Since 1994, Hofbauer and So [9] conjectured the number of limit cycles is at most two for the competitive (LV), the intensive investigations on the limit cycle bifurcation have been triggered, which are generally based on the remarkable result of Hirsch,

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Zeeman [28]: the competitive 3D LV systems have only 33 classes of all possible stable phase portraits and only classes 26-31 of those can have limit cycles.

Though Xiao and Li [25] proved that the number is finite for of the 3D competitive LV system without a heteroclinic polycycle, the maximum number of limit cycles from the interior fixed point in Zeeman's six classes 26-31 remains open up till now (see e.g. [8,15,16,22,27]). At present, four limit cycles is the maximum number given respectively by Yu et al. [27] for class 26 and class 27 by Wang et al. [22] for class 29. As for the non-competitive 3D LV system, the limit cycles bifurcation has been paid little attention to, an example of four possible limit cycles is given by Wang et al. in [21], here we will continue to consider a 3D non-competitive system as follows

$$\dot{x} = \mathrm{diag}(x)A(x-E)', \tag{1.1}$$

where $x = (x_1, x_2, x_3)$, $E = (1,\ 1,\ 1)$ and $()'$ denotes the transpose of a vector, and the interaction matrix

$$A = \begin{pmatrix} 0 & -n & -\nu \\ -h & \lambda & -s \\ -1 & 0 & 0 \end{pmatrix}$$

where n, h, s, ν and λ are real numbers. Obviously, E is the unique positive equilibrium of system (1.1).

In fact, the works on limit cycle bifurcation of systems in $\mathbb{R}^3$ are not unusually seen, especially for the Hopf bifurcation, except on the above 3D LV systems, extensive investigations on chaotic systems have been carried out (see [1,23] and the references therein). Moreover, for the more general higher-dimensional systems, the maximal number of limit cycles which may exist in the vicinity of a Hopf singular point under proper perturbations, i.e. the cyclicity of Hopf bifurcation, is attracting more and more attentions as a very challenging problem. Some good outcomes have also appeared, see for example [26] and references therein.

Furthermore, the problem about the number of limit cycles bifurcated at Hopf point, i.e., the cyclicity, is closely related to center-focus determination. For the center problem on the center manifold, there have been some valid research approaches, such as the averaging theory considered in [2,11], the technique of inverse Jacobi multiplier studied in [3,4], the simplest normal form method given in [20] and the formal first integral method given in [7]. Here, in order to investigate the center problem, we apply the formal series method introduced in [24] to find all the necessary conditions of local integrability, as an upgraded version of the method given in [10] for planar systems, which is also very valid to study the Hopf bifurcation (see e.g. [22,23]).

This paper is organized as follows. In Section 2, after system (1.1) being transformed, the corresponding singular point quantities are computed via the recursion formula derived. In Section 3, the singular point quantities are simplified and the Darboux theory is applied to show the sufficiency of integrability, then the center conditions on the center manifold are determined. In Section 4, the multiple Hopf bifurcations at the equilibrium is investigated for the corresponding system, and it is proved at most five small limit cycles from the positive equilibrium of system (1.1) via Hopf bifurcation. This is an interesting example and also the first case with five possible limit cycles for the cyclicity problem of the 3D LV systems.