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A Class of Schur Convex Functions and Several Geometric Inequalities

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Abstract: Schur convexity, Schur geometrical convexity and Schur harmonic convexity of a class of symmetric functions are investigated. As consequences some known inequalities are generalized. In addition, a class of geometric inequalities involving n-dimensional simplex in n-dimensional Euclidean space E^n and several matrix inequalities are established to show the applications of our results.
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1 Introduction

The Schur convex function was introduced by Schur[1] in 1923 and played a key role in analytic inequalities (see [2–17]). Moreover, the theory of convex functions and Schur convex functions is one of the most important research fields in modern analysis and geometry. The following definitions can be found in many references such as [1–3, 14–15].

Definition 1.1 *Let* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbf{R}^n$ *be two* n*-tuples real numbers.* $\boldsymbol{x}$ *is said to be majorized by* $\boldsymbol{y}$ (*in symbols* $\boldsymbol{x} \prec \boldsymbol{y}$) *if* $\sum\limits_{i=1}^{k} x_{[i]} \le \sum\limits_{i=1}^{k} y_{[i]}$ $(k = 1, 2, \cdots, n-1)$ *and* $\sum\limits_{i=1}^{n} x_{[i]} = \sum\limits_{i=1}^{n} y_{[i]}$, *where* $x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$, $y_{[1]} \ge y_{[2]} \ge \cdots \ge y_{[n]}$ *are sorts of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in descending order.*

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Definition 1.2 *A real-valued function $f : \Omega \subset \mathbf{R}^n \to \mathbf{R}$ is said to be Schur convex on Ω if*

$$\boldsymbol{x} \prec \boldsymbol{y} \quad on\ \Omega \quad \Longrightarrow \quad f(\boldsymbol{x}) \le f(\boldsymbol{y});$$

f is a Schur concave function on Ω if and only if $-f$ is a Schur convex function.

The following lemma is so-called Schur's condition for determining whether or not a given function is Schur convex or Schur concave.

Lemma 1.1[1, P57] *Let $\Omega \in \mathbf{R}^n$ be a symmetric and convex set with nonempty interior, and let $f : \Omega \to \mathbf{R}$ be differentiable in the interior of Ω. Then f is Schur convex (Schur concave) on Ω if and only if f is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \ge 0\ (\le 0), \qquad x \in \Omega^0, \tag{1.1}$$

where Ω^0 is the interior of Ω.

The multiplicatively convex function was recommended by Niculescu[10], which revealed an entire new world of graceful inequalities. Zhang[17] stated the Schur geometrically convex theory as a parallel one to Schur convex theory by defining logarithmic majorization and using multiplicatively convex function.

Definition 1.3[17, P89] *Assume that $\boldsymbol{x}$ and $\boldsymbol{y}$ are two n-tuples of nonnegative numbers. Then the n-tuple $\boldsymbol{x}$ is called logarithmically majorized by $\boldsymbol{y}$ (in symbols $\ln \boldsymbol{x} \prec \ln \boldsymbol{y}$) if*

$$\prod_{i=1}^{k} x_{[i]} \le \prod_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \cdots, n-1, \qquad \prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}.$$

Definition 1.4[17, P107] *Suppose that I is a subinterval of $(0, \infty)$. A function $\varphi : I^n \to (0, \infty)$ is called Schur geometrical convexity if*

$$\ln \boldsymbol{x} \prec \ln \boldsymbol{y} \quad \text{on } I^n \quad \Longrightarrow \quad \varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y});$$

φ is called Schur geometrically concave if $-\varphi$ is Schur geometrically convex.

The following Lemma is basic and plays a fundamental role in the theory of Schur geometrically convex function.

Lemma 1.2[17, P108] *Let $\varphi(\boldsymbol{x}) = \varphi(x_1, x_2, \cdots, x_n)$ be symmetric and have continuously partial derivatives on I^n, where I is a subinterval of $(0, \infty)$. Then $\varphi : I^n \to (0, \infty)$ is Schur geometrically convex (Schur geometrically concave) if and only if*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \ge 0\ (\le 0), \qquad x \in I^n. \tag{1.2}$$

Definition 1.5 *Let $\omega(\boldsymbol{x}) = \omega(x_1, x_2, \cdots, x_n)$ be symmetric and have continuously partial derivatives on I^n, where I is a subinterval of $(0, \infty)$. Then $\omega : I^n \to (0, \infty)$ is a Schur harmonic convex function if*

$$\boldsymbol{x} \prec \boldsymbol{y} \quad \text{on } I^n \quad \Longrightarrow \quad \omega\left(\frac{1}{\boldsymbol{x}}\right) \le \omega\left(\frac{1}{\boldsymbol{y}}\right),$$