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# Certain Fixed Point Theorems and Application to the Fractal Space

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**Abstract:** In this paper, we present some important generalizations of the Banach contraction principle, in which the Lipschitz constant $k$ is replaced by some real-valued control function. For the applications to the fractal space, we obtain the fixed point theorem of the some generalized contraction in the space of fractals.

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## 1 Introduction

The most well known result in the theory of fixed points is Banach's contraction mapping principle. Boyd and Wong[1] and Matkowski[2] proved the theorem of existence of fixed point of mappings in complete metric space. In particular, these theorems extend the result of F. E. Browder's fixed point theorem. They discussed the Banach contraction principle with some generalized contraction conditions and weakened the usual contraction condition. The main idea in the generalization of Banach's contraction theorem was to use the combining of the ideas in the contraction principle.

If all $f_i$ $(i = 1, 2, \cdots, N)$ are Banach contractions, then the mapping

$$F(A) =: \bigcup_{i=1}^{N} F_i(A) \qquad \text{for } A \in H(X)$$

has a unique fixed point $K \in H(X)$, where $H(X)$ denotes the space whose points are the nonempty compact subsets of the complete metric space $(X, d)$. This result was proposed by Barnsley[3]. Máté[4] proved the theorem dealing with finite families of Browder's contractions and the previous works proved theorem dealing with finite (or infinite) families of Matkowski's contractions (see [5–8]), which extended the result of Barnsley.

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The aim of this paper is to obtain the fixed point theorems of the some generalized contractions and present the application of fixed point theorems in the space of fractals. Before we establish the fixed point theorems of the some generalized contraction, we discuss some basic results.

Boyd and Wong[1] and Matkowski[2] proved the following results in the complete metric space.

**Theorem 1.1** [1] (Boyd and Wong's fixed point theorem) *Let $X$ be a complete metric space, and let $f : X \longrightarrow X$ satisfy*

$$d(f(x),\ f(y)) \le \varphi(d(x,\ y)), \qquad x, y \in X,$$

*where $P = \{d(x,\ y) : x, y \in X\}$, $\bar{P}$ is the closure of $P$, $\varphi : \bar{P} \longrightarrow [0, +\infty)$ is upper semi-continuous from right on $\bar{P}$ (not necessarily non-decreasing), and $\varphi(t) < t$ for all $t \in \bar{P}\backslash\{0\}$. Then $f$ has a unique fixed point $p \in X$, and $\lim\limits_{n\to+\infty} f^n(x) = p$, for each $x \in X$.*

**Theorem 1.2** [9] (Matkowski's fixed point theorem) *Let $X$ be a complete metric space and $f : X \longrightarrow X$ be a mapping satisfying that*

$$d(f(x),\ f(y)) \le \varphi(d(x,\ y)), \qquad x, y \in X,$$

*where $\varphi : (0, +\infty) \longrightarrow (0, +\infty)$ is non-decreasing (not necessarily upper semicontinuous function from the right on $(0, +\infty)$) and satisfies $\lim\limits_{n\to+\infty} \varphi^n(t) = 0$ for all $t \in (0, +\infty)$ (necessarily, $\varphi(t) < t$ for all $t \in (0, +\infty)$, see Theorem 1.6 in [10]). Then $f$ has a unique fixed point $p \in X$, and $\lim\limits_{n\to+\infty} f^n(x) = p$ for each $x \in X$.*

These fixed point theorems extended an earlier result of Banach's contraction principle. Jachymski[11] obtained a complete characterization of relations between fixed point theorems of Boyd and Wong[1] and Browder[12].

**Definition 1.1** [11] *We say that a selfmapping $f$ of a metric space $(X,\ d)$ is $\varphi$-contractive if $\varphi$ is a given function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ such that $\varphi(t) < t$ for all $t > 0$, and*

$$d(f(x),\ f(y) \le \varphi(d(x,\ y)), \qquad x, y \in X.$$

**Theorem 1.3** [11] *Let $f$ be a selfmapping of a metric space $(X,\ d)$. The following statements are equivalent:*

(1) *There exists an increasing and right continuous function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ such that $f$ is $\varphi$-contractive (see [12–13]);*

(2) *There exists a continuous function $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\psi(t) < t$ for all $t > 0$ such that (weakly contractive)*

$$d(f(x),\ f(y)) \le d(x,\ y) - \psi(d(x,\ y)), \qquad x, y \in X$$

*(see [14–16]);*

(3) *There exists an upper semicontinuous function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ such that $f$ is $\varphi$-contractive (see [1, 17]);*

(4) *There exists a function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\limsup\limits_{s\to t} \varphi(s) < t$ for all $t > 0$ such that $f$ is $\varphi$-contractive;*