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A $(k, n-k)$ Conjugate Boundary Value Problem with Semipositone Nonlinearity

YAO QING-LIU
(*Department of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing, 210003*)

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Abstract: The existence of positive solution is proved for a $(k, n-k)$ conjugate boundary value problem in which the nonlinearity may make negative values and may be singular with respect to the time variable. The main results of Agarwal *et al.* (Agarwal R P, Grace S R, O'Regan D. Semipositive higher-order differential equations. *Appl. Math. Letters*, 2004, **14**: 201–207) are extended. The basic tools are the Hammerstein integral equation and the Krasnosel'skii's cone expansion-compression technique.

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1 Introduction

Let $n \geq 2$, $1 \leq k \leq n-1$ be two positive integers and $\lambda > 0$ be a positive parameter. In this paper, we study the existence of positive solution to the following nonlinear $(k, n-k)$ conjugate boundary value problem:

$$\text{(P)} \qquad \begin{cases} (-1)^{n-p}u^{(n)}(t) = \lambda f(t, u(t)), & 0 < t < 1, \\ u^{(i)}(0) = 0, \quad u^{(j)}(1) = 0, & 0 \leq i \leq k-1, \ 0 \leq j \leq n-k-1. \end{cases}$$

The solution u^* of the problem (P) is called positive if $u^*(t) > 0$ for $0 < t < 1$.

For the function $f(t, x)$, we use the following assumptions:

(A1) $f : (0, 1) \times [0, +\infty) \to (-\infty, +\infty)$ is continuous.

(A2) There exists a nonnegative function $h \in L^1[0, 1] \cap C(0, 1)$ such that

$$f(t, x) + h(t) \geq 0, \qquad (t, x) \in (0, 1) \times [0, +\infty).$$

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E-mail address: yaoqingliu2002@hotmail.com (Yao Q L).

(A3) For each $r > 0$, there exists a nonnegative function $j_r \in L^1[0, 1] \cap C(0, 1)$ such that

$$f(t, x) + h(t) \le j_r(t), \qquad (t, x) \in (0, 1) \times [0, r].$$

The assumptions (A2) and (A3) show that $f(t, x)$ may be singular at $t = 0$ and $t = 1$, and may not have any numerical lower bound. Therefore, the problem (P) is singular and semipositone. The problems of this type arise naturally in chemical reactor theory, see [1].

In applications, one is interested in showing the existence of positive solution for some λ. When $h(t) \equiv M \ge 0$, the problem (P) has been frequently investigated in recent years, for example, see [2–9] and the references therein.

In 2004, Agarwal *et al.*[8] established the following existence theorem of positive solution:

Theorem 1.1 ([8], Theorem 2.3) *Suppose that the following conditions are satisfied*:

(a1) $f : [0, 1] \times [0, +\infty) \to (-\infty, +\infty)$ *is continuous and there exists a constant* $M > 0$ *such that* $f(t, x) + M \ge 0$ *for any* $(t, x) \in [0, 1] \times [0, +\infty)$;

(a2) *There exists a continuous and nondecreasing function* $\zeta : [0, +\infty) \to [0, +\infty)$ *such that*

$$\zeta(x) > 0, \qquad 0 < x < +\infty,$$

and

$$f(t, x) + M \le \zeta(x), \qquad (t, x) \in [0, 1] \times [0, +\infty);$$

(a3) *There exists a positive number* $r_1 \ge \dfrac{\lambda M}{n!}$ *such that*

$$\lambda\zeta(r_1) \max_{0\le t\le 1} \int_0^1 G(t, s)\mathrm{d}s \le r_1;$$

(a4) *There exist a* δ *with* $0 < \delta < \dfrac{1}{2}$ *and a continuous and nondecreasing function* $\xi : (0, +\infty) \to (0, +\infty)$ *such that*

$$f(t, x) + M \ge \xi(x), \qquad (t, x) \in [\delta, 1 - \delta] \times (0, +\infty);$$

(a5) *There exists an* ε *with*

$$0 < \varepsilon \le 1 - \frac{\lambda M}{n! r_2}, \qquad r_2 > r_1$$

such that

$$\lambda\xi(\varepsilon\theta r_2) \max_{0\le t\le 1} \int_\delta^{1-\delta} G(t, s)\mathrm{d}s \ge r_2,$$

where

$$\theta = \begin{cases} \delta^k(1-\delta)^{n-k}, & n \le 2k; \\ \delta^{n-k}(1-\delta)^k, & n \ge 2k. \end{cases}$$

Then the problem (P) *has at least one positive solution* $u^* \in C^{n-1}[0, 1] \cap C^n(0, 1)$.

In Theorem 1.1, $G(t, s)$ is the Green function of the problem (P) with $f(t, x) \equiv 0$. For the expression of $G(t, s)$, see Section 2. The function $h(t) \equiv M$ is a constant and the nonlinearity $f(t, x)$ is continuous on $[0, 1] \times [0, +\infty)$.