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Certain Pair of Derivations on a Triangular Algebra

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Abstract: By using properties of triangular algebra, we prove that if derivations D and G on a triangular algebra $\mathcal{T}$ satisfy certain generalized identities, then both D and G are zero mappings. As a corollary we get that if D and G are cocentralizing on $\mathcal{T}$, then both D and G are zero mappings.

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1 Introduction

Let R be a commutative ring with identity, A and B be unital algebras over R, and M be a nonzero (A, B)-bimodule. Then we define

$$\begin{bmatrix} A & M \\ 0 & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \middle| a \in A,\ b \in B,\ m \in M \right\}$$

to be an associative algebra under matrix-like addition and matrix-like multiplication. An algebra $\mathcal{T}$ is called a triangular algebra if there exist R-algebras A, B and nonzero (A, B)-bimodule M such that $\mathcal{T}$ is (algebraically) isomorphic to $\begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ under matrix-like addition and matrix-like multiplication. Usually, we denote a triangular algebra by

$$\mathcal{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}.$$

This kind of algebra was first introduced by Chase[1]. Harada[2] referred to the triangular algebras as generalized triangular matrix rings. The definition of triangular algebra is somewhat formal and hence they are said to be formal triangular matrix algebras in the case of noncommutative algebras (see [3]).

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Throughout this paper, we set $[y, x] = [y, x]_1 = yx - xy$; and for $k > 1$, we set $[y, x]_k = [[y, x]_{k-1}, x]$. An R-linear map $L : A \to A$ is said to satisfy the Engel condition (or to be k-commuting, according to [4]) on A if $[L(x), x]_k = 0$ for all $x \in A$. In particular, if $[L(x), x] = 0$ for all $x \in A$, then L is said to be commuting on A; if $[L(x), x] \in Z(A)$ for all $x \in A$, then L is said to be centralizing on A, where $Z(A)$ denotes the center of A.

Let R be a ring. A map $d : R \to R$ is called a derivation if

$$d(ab) = d(a)b + ad(b), \qquad a, b \in R.$$

The study of derivations on prime rings was initiated by Posner[5]. He proved that if R is a prime ring and d is a nonzero commuting derivation on R, i.e., $d(x)x - xd(x) = 0$ for all $x \in R$, then R is commutative. A number of authors have extended this theorem in several ways. Two maps d and g are called cocentralizing on a ring R, if $d(x)x - xg(x) \in Z(R)$ for all $x \in R$, where $Z(R)$ denotes the center of R. Brešar[6] showed that if R is a prime ring, then I is a nonzero left ideal of R; and if derivations $d \neq 0$ and g are cocentralizing on I, then R is commutative. Vukman[7] showed that R is commutative if either $[D(x), x]_2 = 0$ for all $x \in R$ and $\mathrm{char}(R) \neq 2$ or if $[D(x), x]_2$ is central for all $x \in R$ and $\mathrm{char}(R) \neq 2, 3$. Lanski[8] generalized Posner's theorem by considering a more general case where a derivation satisfies the Engel condition on a prime ring.

In recent years, some authors also work on derivations on triangular algebras. Xie and Cao[9], Li and Xu[10] gave the exact form of derivations and Jordan derivations on triangular algebras, respectively. Zhang and Yu[11] showed that every Jordan derivation on a triangular algebra is a derivation. Han and Wei[12] declared that (α, β)-derivation, Jordan (α, β)-derivation and Jordan triple (α, β)-derivation are equivalent on triangular algebras.

Du and Wang[4] characterized maps which satisfy the Engel condition on a triangular algebra.

In this paper, we work on derivations on a triangular algebra. We discuss these cases where a pair of derivations of a triangular algebra satisfy certain generalized identities, that is,

$$\begin{aligned} &D(X^m)X^n - X^n G(X^m) = 0,\\ &[D(X), X]_k X - X[G(X), X]_k = 0,\\ &[D(X)X - XG(X), X]_k = 0, \end{aligned}$$

where m, n, k are positive integers. As a corollary we get that if D and G are cocentralizing on a triangular algebra, then $D = G = 0$.

Throughout this paper, R always denotes a commutative ring with identity. Let A and B be unital algebras over R and M be a unital (A, B)-bimodule, which is faithful as a left A-module and also as a right B-module. Let $\boldsymbol{\mathcal{T}} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be the triangular algebra consisting of A, B and M. Then $\boldsymbol{\mathcal{T}}$ is an associative and noncommutative R-algebra with identity $\boldsymbol{I} = \begin{bmatrix} 1_A & 0 \\ 0 & 1_B \end{bmatrix}$. We always set D and G be derivations on $\boldsymbol{\mathcal{T}}$. Then by using the result of Xie and Cao[9], we know that there are derivations d_A, g_A on algebra A, d_B, g_B on algebra B, additive mappings f, h on the nonzero module M and $u, v \in M$ such that for