Communications
in
Mathematical
Research
28(3)(2012), 265–274

Non-linear Invertible Maps that Preserve Staircase Subalgebras*

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Communicated by Du Xian-kun

Abstract: Let $\mathfrak{g}$ be the general linear Lie algebra consisting of all $n \times n$ matrices over a field F and with the usual bracket operation $[x, y] = xy - yx$. An invertible map $\varphi : \mathfrak{g} \to \mathfrak{g}$ is said to preserve staircase subalgebras if it maps every staircase subalgebra to some staircase subalgebra of the same dimension. In this paper, we devote to giving an explicit description on the invertible maps on $\mathfrak{g}$ that preserve staircase subalgebras.

Key words: general linear Lie algebra, staircase subalgebra, non-linear map

2000 MR subject classification: 15A04, 15A27, 17B30

Document code: A

Article ID: 1674-5647(2012)03-0265-10

1 Introduction

Let F be a field, n a positive integer (≥ 2), V the vector space of column vectors of length n having entries in F, and e_i ($i = 1, 2, \cdots, n$) the unit vector with 1 the only nonzero entry in the i-th position. Let $\mathfrak{g}$ be the general linear Lie algebra of all $n \times n$ matrices over F with the usual bracket operation $[x, y] = xy - yx$, and $\mathfrak{b}$ (resp., $\mathfrak{d}$) the subalgebra of $\mathfrak{g}$ consisting of all upper triangular (resp. diagonal) matrices over F. For $x \in \mathfrak{g}$, we denote by x' the transpose of x. Denote by e_{ij} the standard matrix unit whose (i, j)-entry is 1 and elsewhere is 0. All e_{ij} ($i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$) form a basis of $\mathfrak{g}$. Thus any element of $\mathfrak{g}$ can be uniquely written as

$$x = \sum_{1 \leq i, j \leq n} x_{ij} e_{ij}, \qquad x_{ij} \in F.$$

*Received date: March 22, 2010.

Foundation item: The NSF (11126121) of China, Ph.D. Fund (B2010-93) of Henan Polytechnic University, Natural Science Research Program (112300410120) of Science and Technology Department of Henan Province and Natural Science Research Program (2011B110016) of Education Department of Henan Province.

In general, a flag on V is a nested sequence of non-zero subspaces of V which terminates in V and has no repeated terms, in other words, a flag is a sequence $\{W_1, W_2, \cdots, W_r\}$ of subspaces of V, where

$$0 \subset W_1 \subset W_2 \subset \cdots \subset W_{r-1} \subset W_r = V.$$

A complete flag is simply a flag for which $r = n$. The standard complete flag is defined to be $\{V_1, V_2, \cdots, V_n\}$, where

$$V_i = Fe_1 \oplus Fe_2 \oplus \cdots \oplus Fe_i, \qquad i = 1, 2, \cdots, n.$$

By a standard flag we mean a flag $\{W_1, W_2, \cdots, W_r\}$ in which each W_i is equal to some V_j. It is easy to see that there are 2^{n-1} such standard flags. There is a natural action of $\mathfrak{g}$ on the set of flags on V, namely, if $\{W_1, \cdots, W_r\}$ is a flag and $x \in \mathfrak{g}$, we define

$$x(W_1, \cdots, W_r) = (xW_1, \cdots, xW_r),$$

where we view x as a linear transformation on V by multiplying each element in V from the left. We say that a subalgebra of $\mathfrak{g}$ is a parabolic subalgebra if it is the stabilizer of some flag on V. A staircase subalgebra is just a special parabolic subalgebra that is the stabilizer of some standard flag on V. The terminology staircase subalgebra arises from the appearance of the matrices in these subalgebras. By a similar method as on Page 53 in [1], we can prove that

(1) The parabolic subalgebras of $\mathfrak{g}$ are exactly the conjugates of the staircase subalgebras;

(2) The subalgebras of $\mathfrak{g}$ containing $\mathfrak{b}$ are exactly the staircase subalgebras.

Suppose that $\{W_1, W_2, \cdots, W_r\}$ is a standard flag with

$$\dim W_1 = n_1, \quad \dim W_2 = n_1 + n_2, \quad \cdots, \quad \dim W_r = n_1 + n_2 + \cdots + n_r.$$

Then the staircase subalgebra, being the stabilizer of $\{W_1, W_2, \cdots, W_r\}$ in $\mathfrak{g}$, consists of all the block upper triangular matrices of the form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ 0 & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_{rr} \end{pmatrix}, \qquad A_{ij} \in M_{n_i \times n_j}(F).$$

It is easy to see that the above staircase subalgebra is completely determined by the sequence $\{n_1, n_2, \cdots, n_r\}$, so we denote it by $S_{(n_1, n_2, \cdots, n_r)}$. On the definition of a staircase subalgebra of $\mathfrak{g}$, we mainly consult [1], where the staircase subgroup of a general linear group is defined.

More recently, some authors have devoted to determining ad-nilpotent ideals of certain special subalgebras of simple Lie algebras. Cellini and Papi[2−3] determined all ad-nilpotent ideals of a Borel subalgebra of a simple Lie algebra over the complex field $\mathbf{C}$, and Righi[4] extented the results of Cellini and Papi to a parabolic subalgebra of a simple Lie algebra. Some other authors were interested in describing invertible transformations on linear Lie algebras that preserve certain special subalgebras. For instance, Radjavi and Šemrl[5] gave an explicit description on the non-linear invertible transformations on the general linear algebra over the complex field that preserve solvable subalgebras. It is known that a solvable subalgebra of $\mathfrak{g}$ can be viewed (up to an isomorphism) as a subalgebra of $\mathfrak{g}$ that is contained in