COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
28(1)(2012), 51–64

Quasi-periodic Solutions of the General Nonlinear Beam Equations*

GAO YI-XIAN[1,2]

(*1. College of Mathematics and Statistics, Northeast Normal University, Changchun, 130024*)

(*2. Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of Education, Jilin University, Changchun, 130012*)

Communicated by Li Yong

Abstract: In this paper, one-dimensional (1D) nonlinear beam equations of the form

$$u_{tt} - u_{xx} + u_{xxxx} + mu = f(u)$$

with Dirichlet boundary conditions are considered, where the nonlinearity f is an analytic, odd function and $f(u) = O(u^3)$. It is proved that for all $m \in (0, M^*] \subset \mathbf{R}$ (M^* is a fixed large number), but a set of small Lebesgue measure, the above equations admit small-amplitude quasi-periodic solutions corresponding to finite dimensional invariant tori for an associated infinite dimensional dynamical system. The proof is based on an infinite dimensional KAM theory and a partial Birkhoff normal form technique.

Key words: beam equation, KAM theorem, quasi-periodic solution, partial Birkhoff normal form

2000 MR subject classification: 37K55

Document code: A

Article ID: 1674-5647(2012)01-0051-14

1 Introduction and Main Result

Consider the general nonlinear beam equations of the form

$$u_{tt} - u_{xx} + u_{xxxx} + mu = f(u) \tag{1.1}$$

on the finite x-interval $[0, \pi]$ with Dirichlet boundary conditions

$$u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0, \tag{1.2}$$

*Received date: Dec. 2, 2009.

Foundation item: The NSF (11001042) of China, the SRFDP Grant (20100043120001) and FRFCU Grant (09QNJJ002).

where the parameter $m \in (0, M^*] \subset \mathbf{R}$, the nonlinearity f is assumed to be real analytic in u and of the form

$$f(u) = au^3 + \sum_{n\geq 5} f_n u^n, \qquad a \neq 0. \tag{1.3}$$

We study the equations of the form (1.1) as a Hamiltonian system on

$$\mathcal{P} = H_0^1([0,\pi]) \times L^2([0,\pi])$$

with coordinates u and $v = u_t$. Then the Hamiltonian is

$$H = \frac{1}{2}\langle v,\, v\rangle + \frac{1}{2}\langle Au,\, u\rangle + \int_0^\pi g(u)\mathrm{d}x, \tag{1.4}$$

where

$$A = \frac{\mathrm{d}^4}{\mathrm{d}x^4} - \frac{\mathrm{d}^2}{\mathrm{d}x^2} + m, \qquad g = \int_0 -f(s)\mathrm{d}s, \tag{1.5}$$

and $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the usual scalar product in L^2. Then (1.1) can be written in the form

$$u_t = \frac{\partial H}{\partial v} = v, \qquad v_t = -\frac{\partial H}{\partial u} = -Au - f(u). \tag{1.6}$$

Let

$$\phi_j(x) = \sqrt{\frac{2}{\pi}}\sin jx, \quad \lambda_j = \sqrt{j^4 + j^2 + m}, \qquad j = 1, 2, \cdots$$

be the basic modes and frequencies of the linear equation

$$u_{tt} - u_{xx} + u_{xxxx} + mu = 0$$

with Dirichlet boundary conditions (1.2). Then every solution of the linear equation is the superposition of their harmonic oscillations and of the form

$$u(t,x) = \sum_{j\geq 1} q_j(t)\phi_j(x), \qquad q_j(t) = \sqrt{I_j}\cos(\lambda_j t + \theta_j),$$

with amplitudes $I_j \geq 0$ and initial phases θ_j. The motions are periodic or quasi-periodic, respectively, depending on whether one or finitely many eigenfunctions are excited. In particular, for every choice

$$J = \{j_1 < j_2 < \cdots < j_n\} \subset \mathbf{N}$$

of finitely many modes there exists an invariant $2n$-dimensional linear subspace E_J which is completely foliated into rotational tori with frequencies $\lambda_{j_1}, \cdots, \lambda_{j_n}$:

$$E_J = \{(u,v) = (q_1\phi_{j_1} + \cdots + q_n\phi_{j_n},\, p_1\phi_{j_1} + \cdots + p_n\phi_{j_n})\} = \bigcup_{I\in\overline{P^n}} \mathcal{T}_J(I),$$

where

$$P^n = \{I \in \mathbf{R}^n : I_j > 0,\ 1 \leq j \leq n\}$$

is the positive quadrant in $\mathbf{R}^n$ and

$$\mathcal{T}_J(I) = \{(u,v) : q_j^2 + \lambda_j^{-2}p_j^2 = I_j,\ 1 \leq j \leq n\},$$

by using the above representations of u and v. In addition, such a torus is linearly stable, and all solutions have zero Lyapunov exponents.

Upon restoration of the nonlinearity f, we show that there exist a Cantor set $\mathcal{O} \subset P^n$, a family of n-tori

$$\mathcal{T}_J[\mathcal{O}] = \bigcup_{I\in\mathcal{O}} \mathcal{T}_J(I) \subset E_J \qquad \text{over } \mathcal{O},$$