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Characterizing Continuous dcpos by Liminf Convergence of Filters*

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Abstract: It is proved in this note that, under a mild assumption, a dcpo L is continuous if and only if the liminf convergence on L is topological.

Key words: continuous dcpo, meet continuous dcpo, convergence space, limit space, pretopological space, S-convergence, liminf convergence

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1 Introduction

The interaction between order theoretic and topological properties is an important topic in order theory (see [1]–[6]). A well-known result is that a dcpo L is continuous if and only if the S-convergence on L is topological. It is proved that in a continuous dcpo the liminf convergence of nets is topological and agrees with convergence in Lawson topology (see [1], Theorem III-3.17). The purpose of this note is to show that the converse is also true under a mild assumption. Precisely, for a strong meet continuous dcpo L, the liminf convergence of nets on L is topological if and only if L is continuous.

2 Convergence Spaces

This section recalls some basic notions of convergence spaces. For each set X, let $F(X)$ denote the set of filters on X. For each $x \in X$, $\hat{x} = \{A \subseteq X \mid x \in A\}$ denotes the principal filter generated by $\{x\}$.

A convergence structure on X is a subset $\mathcal{T} \subseteq F(X) \times X$ subject to the following conditions:

(1) $(\hat{x}, x) \in \mathcal{T}$ for all $x \in X$;

(2) $(\mathcal{F}, x) \in \mathcal{T}$, $\mathcal{F} \subseteq \mathcal{G} \Longrightarrow (\mathcal{G}, x) \in \mathcal{T}$.

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The pair $(X, \mathcal{T})$ is called a convergence space. If $(\mathcal{F},\ x) \in \mathcal{T}$, we also write $\mathcal{F} \xrightarrow{\mathcal{T}} x$ or simply $\mathcal{F} \to x$ if no confusion will arise. A function $f : X \longrightarrow Y$ between two convergence spaces is continuous if $\mathcal{F} \to x$ implies that $f(\mathcal{F}) \to f(x)$, where $f(\mathcal{F})$ is the filter on Y generated by $\{f(A) \mid A \in \mathcal{F}\}$.

The category of convergence spaces and continuous functions is denoted by Conv. It is well-known that Conv is topological over the category of sets (see [7]), and hence it is complete and cocomplete. In particular, the product $(X_1 \times X_2,\ \mathcal{T}_1 \times \mathcal{T}_2)$ of two convergence spaces $(X_1,\ \mathcal{T}_1)$ and $(X_2,\ \mathcal{T}_2)$ is given as follows: a filter $\mathcal{F}$ on $X_1 \times X_2$ converges to $(x_1,\ x_2)$ with respect to $\mathcal{T}_1 \times \mathcal{T}_2$ if and only if there exist $(\mathcal{F}_1, x_1) \in \mathcal{T}_1$ and $(\mathcal{F}_2, x_2) \in \mathcal{T}_2$ such that $\{A \times B \mid A \in \mathcal{F}_1,\ B \in \mathcal{F}_2\} \subseteq \mathcal{F}$. Note that here we use the symbol $\mathcal{T}_1 \times \mathcal{T}_2$ to denote the product convergence structure, not the cartesian product of $\mathcal{T}_1$ and $\mathcal{T}_2$ (as sets), which will cause no confusion.

Given a topological space X, $\mathcal{T} = \{(\mathcal{F}, x) \mid \mathcal{F} \to x\}$ is a convergence structure on X. This correspondence is indeed a functorial embedding of the category Top of topological spaces in Conv.

Definition 2.1 *Let $(X,\ \mathcal{T})$ be a convergence space.*

(1) $(X,\ \mathcal{T})$ *is called a Kent convergence space if*

(K) $\mathcal{F} \to x \Longrightarrow \mathcal{F} \cap \hat{x} \to x.$

(2) $(X,\ \mathcal{T})$ *is called a limit space if*

(Lim) $\mathcal{F} \to x,\ \mathcal{G} \to x \Longrightarrow \mathcal{F} \cap \mathcal{G} \to x.$

(3) $(X,\ \mathcal{T})$ *is called a pretopological space if*

(PrT) $\mathcal{F}_j \to x,\ \forall t \in J \Longrightarrow \bigcap\limits_{j \in J} \mathcal{F}_j \to x.$

(4) $(X,\ \mathcal{T})$ *is called topological if $\mathcal{T}$ is generated by some topology on X.*

It is clear that every topological convergence space is pretopological, every pretopological space is a limit space, and every limit space is a Kent convergence space. The full subcategories consisting of Kent convergence spaces, limit spaces, and pretopological spaces are denoted respectively by KConv, Lim, and PrTop. Then

$$\text{Top} \subset \text{PrTop} \subset \text{Lim} \subset \text{KConv} \subset \text{Conv}.$$

It is well known (e.g., [8]) that every category in the above diagram contains the preceding ones as reflective subcategories.

3 S-convergence

A dcpo is a partially ordered set L such that every directed set $D \subseteq L$ has a supremum. We write DCPO for the category of dcpos and functions that preserve directed suprema.

Definition 3.1 *Let $\mathcal{F}$ be a filter on a dcpo L.*

(1) *$z \in L$ is an eventual lower bound of $\mathcal{F}$ if $\uparrow z \in \mathcal{F}$.*

(2) *$\mathcal{F}$ S-converges to $x \in L$ (in symbols, $\mathcal{F} \to_S x$) if there exists a directed set D contained in the eventual lower bounds of $\mathcal{F}$ with $x \leq \sup D$.*