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A MODIFIED PRIMAL-DUAL WEAK GALERKIN FINITE ELEMENT METHOD FOR SECOND ORDER ELLIPTIC EQUATIONS IN NON-DIVERGENCE FORM

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Abstract. A modified primal-dual weak Galerkin (M-PDWG) finite element method is designed for the second order elliptic equation in non-divergence form. Compared with the existing PDWG methods proposed in [6], the system of equations resulting from the M-PDWG scheme could be equivalently simplified into one equation involving only the primal variable by eliminating the dual variable (Lagrange multiplier). The resulting simplified system thus has significantly fewer degrees of freedom than the one resulting from existing PDWG scheme. Optimal order error estimates are derived for the numerical approximations in the discrete H^2-norm, H^1-norm and L^2-norm respectively. Extensive numerical results are demonstrated for both the smooth and non-smooth coefficients on convex and non-convex domains to verify the accuracy of the theory developed in this paper.

Key words. Primal-dual, weak Galerkin, finite element methods, non-divergence form, Cordès condition, polyhedral meshes.

1. Introduction

In this paper, we consider the second order elliptic equation in non-divergence form which seeks an unknown function $u = u(x)$ such that

$$\begin{aligned} \sum_{i,j=1}^{d} a_{ij}\partial^2_{ij} u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is an open bounded domain with Lipschitz continuous boundary $\partial\Omega$, the load function $f \in L^2(\Omega)$, and the coefficient tensor $a = (a_{ij})_{d\times d} \in [L^\infty(\Omega)]^{d\times d}$ is symmetric, uniformly bounded and positive definite in the sense that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \xi^T \xi \leq \xi^T a \xi \leq C_2 \xi^T \xi, \qquad \forall \xi \in \mathbb{R}^d, \ x \in \Omega. \tag{2}$$

For the simplicity of notation, denote by $\mathcal{L} := \sum_{i,j=1}^{d} a_{ij}\partial^2_{ij}$ the second order partial differential operator.

The second order elliptic problem in non-divergence form arises in various applications such as probability and stochastic processes [2]. This type of problem also plays an important role in the research of fully nonlinear partial differential equations in conjunction with linearization techniques (e.g., the Newton's iterative method) [1, 3]. In such applications, the coefficient tensor $a(x)$ is often hardly smooth. Therefore, it is crucial to develop effective numerical methods for the model problem (1) with nonsmooth coefficient tensor. Readers are referred to [6] for more details of recent work developed for the model problem (1) . f The goal of

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this paper is to develop a modified primal-dual weak Galerkin (M-PDWG) scheme for the second order elliptic problem in nondivergence form (1), which is different from and advantageous over the one proposed in [6]. The system of equations arising from the M-PDWG scheme could be equivalently simplified into one equation by eliminating its dual variable (Lagrange multiplier). The simplified system involves only the primal variable and thus has significantly fewer degrees of freedom compared to the PDWG scheme proposed in [6]. The main contribution of the present paper is that the numerical scheme admists a simplified form with reduced computational complexity. Our theory for the M-PDWG method is based on two assumptions: (1) the H^2-regularity of the exact solution of the model problem (1); and (2) the coefficient tensor $a(x)$ is piecewise continuous and satisfies the uniform ellipticity condition (2). Optimal order error estimates are established for the primal variable in a discrete H^2-norm and for the dual variable in the L^2-norm. Moreover, the convergence theory is derived for the primal variable in the H^1 norm and L^2 norm under some smoothness assumptions for the coefficient tensor $a(x)$. Numerical examples are presented to illustrate the accuracy of the theory developed for the M-PDWG method.

The paper is organized as follows. In Section 2, we present the weak formulation for the model problem (1). Section 3 is devoted to a review of weak second order differential operator and its discretization. In Section 4, we describe the M-PDWG finite element method for the model problem (1). Section 5 presents a simplified system resulting form the M-PDWG method proposed in Section 4. Section 6 is devoted to a stability analysis for the M-PDWG scheme. Section 7 presents the error equations for the numerical scheme. In Section 8, we derive an optimal order error estimate for the M-PDWG method in a discrete H^2 norm. Section 9 establishes some error estimates in the usual H^1 norm and L^2 norm for the primal variable. In Section 10, the numerical experiments are presented for the M-PDWG scheme for smooth and non-smooth coefficient tensor $a(x)$ on convex and non-convex domains.

2. Variational Formulations

We shall briefly review the weak formulation of the second order elliptic model problem (1) in non-divergence form [6].

Theorem 2.1. [4] *Assume (1) $\Omega \subset \mathbb{R}^d$ is a bounded convex domain; (2) the coefficient tensor $a = (a_{ij}) \in [L^\infty(\Omega)]^{d\times d}$ satisfies the ellipticity condition (2); and (3) the Cordès condition holds true; i.e., there exists an $\varepsilon \in (0, 1]$ such that*

$$\frac{\sum_{i,j=1}^d a_{ij}^2}{(\sum_{i=1}^d a_{ii})^2} \le \frac{1}{d-1+\varepsilon} \qquad \text{in } \Omega. \tag{3}$$

There exists a unique strong solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of the model problem (1) satisfying

$$\|u\|_2 \le C\|f\|_0, \tag{4}$$

for any given $f \in L^2(\Omega)$, where C is a constant depending on d, the diameter of Ω, C_1, C_2 and ε.

Throughout this paper, we assume the model problem (1) has a unique strong solution in $H^2(\Omega) \cap H_0^1(\Omega)$ with a priori estimate (4).