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AN EFFECTIVE ALGORITHM FOR COMPUTING FRACTIONAL DERIVATIVES AND APPLICATION TO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In recent years, fractional differential equations have been extensively applied to model various complex dynamic systems. The studies on highly accurate and efficient numerical methods for fractional differential equations have become necessary. In this paper, an effective recurrence algorithm for computing both the fractional Riemann-Liouville and Caputo derivatives is proposed, and then spectral collocation methods based on the algorithm are investigated for solving fractional differential equations. By the recurrence method, the numerical stability with respect to N, the number of collocation points, can be improved remarkably in comparison with direct algorithm. Its robustness ensures that a highly accurate spectral collocation method can be applied widely to various fractional differential equations.

Key words. Riemann-Liouville derivative, Caputo fractional derivative, Riesz fractional derivative, spectral collocation method, fractional differentiation matrix.

1. Introduction

Fractional differential equations (FDEs) have been applied widely in many recent studies in applied mathematics, theoretical physics and mechanics, biology, and economics [22, 23, 30, 31]. The fractional derivative is a powerful tool to describe complex systems that have long memory and long-range spatial interactions. In general, however, numerical methods for fractional derivatives and fractional differential equations suffer from heavy costs of computing due to their nature of non-locality. Therefore, numerical study on fractional differential equations by highly accurate and efficient methods is an imperative task. The spectral method, which is suitable for the discretization of the fractional derivative as a global scheme, has begun to draw more and more attentions from scientific researchers [1, 14, 16, 20, 34, 35, 38, 39, 40].

However, one has to overcome two main difficulties for the spectral method dealing with FDEs: one is the computation of fractional derivatives, and the other is the singularity of the solution of FDEs. Indeed, both difficulties are related with the choice of basis functions the spectral method adopted. Today, it is clear that the use of fractional Jacobi functions (also are called the generalized Jacobi functions) as basis functions is more suitable to deal with the singularity of the solution [3, 6, 7, 10, 32, 33]. Nevertheless, the classical polynomial basis is convenient for the computations of FDEs and also to the analysis of spectral approximation [11, 15, 29]. Moreover, using the polynomials as basis function is still highly accurate compared with the other numerical methods, such as finite difference method and finite element method [15]. It is worthwhile to note that some authors engaged in high order methods for the discretization of fractional derivatives, see [17, 18] and the recent works [8, 19] for example.

In this paper, we are concerned with developing an effective algorithm for computing fractional derivatives. The classical polynomials are still adopted here due

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to the consideration that the singularity near boundary can be overcome by the spectral element method based on polynomials, see the recent paper [21] for details. Several recurrence methods are proposed for the computation of the left- and right- Riemann-Liouville fractional derivatives and the left- and right- Caputo fractional derivatives here. Especially, we compare the stability of our method to one of the direct methods [4, 5, 11, 26]. Then, some applications based on the spectral collocation method are presented. Meanwhile, a comparison with the collocation method based on fractional Jacobi functions [32, 33] is performed.

In [15], Li, Zeng and Liu developed a recurrence method to compute fractional integrals and derivatives. Utilizing the three-term recurrence relation and the property of Jacobi polynomials, the authors established a recurrence scheme for the computation of fractional derivatives. In [37] the author presented a spectral/spectral collocation method by using this recurrence method for solving the space fractional diffusion equation. Anyway, the recurrence algorithm is worth further developing for the spectral collocation method due to its high efficiency and accuracy.

We shall take a different route to compute the differentiation matrix in this paper. The main idea comes from the fact that if $f \in P_n^{\alpha,\beta}(x)$, then it implies $\partial_x f \in P_{n-1}^{\alpha+1,\beta+1}(x)$, where $P_n^{\alpha,\beta}(x)$ designates the class of Jacobi polynomials. Therefore, for the Chebyshev polynomials of the first kind with $\alpha = \beta = -\frac{1}{2}$, their derivatives of the first order are the Chebyshev polynomials of the second kind with $\alpha = \beta = \frac{1}{2}$. Thus, some properties of the Chebyshev polynomials of the second kind can be employed, and a simplified and effective recurrence algorithm for the computation of fractional derivatives of the Chebyshev polynomials of the first kind is then derived.

The paper is arranged as follows. In Section 2 we introduce the series expansion of the Jacobi polynomials in detail starting from an eigenvalue problem, and present the direct method for computing the fractional derivatives. The derivations of the recurrence algorithms are presented in Section 3. In Section 4, the fractional differentiation matrices are investigated, and then the approximated errors are proposed, and several examples are presented to illustrate the stability of our method for large polynomial degree. Some applications of our method are considered in Section 5. Here, we mainly consider the multi-term fractional equations, time-space fractional diffusion equations, and the Riesz fractional diffusion equations. We also consider the non-smooth problem, and a comparison with the corrected backward formulae is carried out in this section. Finally, some remarks and conclusions are presented in Section 6.

2. Preliminaries

At first, we recall a fundamental result about the singular eigenvalue problem and some useful analytical formulations of Jacobi polynomials for computation of fractional derivatives(see also [5, 11, 15, 24]). Consider the following eigenvalue problem

$$(w\varphi y_n')'(x) = \lambda_n w(x) y_n(x). \tag{1}$$

where the weight function satisfies the Pearson equation (see [13] for details)

$$(w\varphi)'(x) = w(x)\psi(x)$$

and $\varphi(x) = x^2 + 2rx + s$, $\psi(x) = 2px + q$, and the eigenvalue $\lambda_n = n(n-1+2p)$.

Lemma 2.1 ([12])**.** *Let $\alpha > -1, \beta > -1$, and r, s, p, q satisfy*

$$2r = -a - b,\ s = ab,\ 2p = \alpha + \beta + 2,\ q = -a(\beta+1) - b(\alpha+1).$$