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Weak Dissipative Structure for Compressible Navier-Stokes Equations*

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Abstract: This paper concerns the Cauchy problem for compressible Navier-Stokes equations. The weak dissipative structure is explored and a new proof for the classical solutions are shown to exist globally in time if the initial data is sufficiently small.

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1 Introduction

The Navier-Stokes equations are the foundation of fluid mechanics and describe the motion of a fluid in some domain of $\mathbf{R}^n$ ($n \leq 3$). The equations are to be solved for an unknown velocity vector $u(x,t) = \big(u_i(x,t)\big)_{1\leq i\leq n} \in \mathbf{R}^n$ and density $\rho(t,x) \in \mathbf{R}$, defined for spatial position $x \in \mathbf{R}^n$ and time $t \geq 0$. We restrict attention here to isentropic compressible fluids filling all of $\mathbf{R}^n$. The Navier-Stokes equations are then given by

$$\begin{cases} \rho_t + u\cdot\nabla\rho + \rho\nabla\cdot u = 0, \\ \rho(u_t + u\cdot\nabla u) = \mu\Delta u + (\lambda+\mu)\nabla\nabla\cdot u - \nabla p, \end{cases} \quad x \in \mathbf{R}^n, \quad t \geq 0, \tag{1.1}$$

where $p = p(\rho)$ is the pressure, μ (the shear viscosity) and λ (the second viscosity) are both constants with $\mu > 0$ and $\lambda + \mu > 0$. For more physical backgrounds, see [1] and [2].

In this paper, we prove that the Cauchy problem of system (1.1) admits a unique classical solution under the assumption that the initial data is small enough. Similar results are studied in [3]. However, our proof is different from that and is due to the weak dissipative structure of Navier-Stokes equations revealed in our main theorem.

To explore the weak dissipative mechanism governing the compressible Navier-Stokes equations, we need use the basic energy law which plays an important role in our proof.

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Let us below derive the basic energy law of the compressible Navier-Stokes equations by an energetic variation method.

Before introducing the action functional, let us recall some basic mechanical evolutions. The dynamics of classical mechanical problems can be described by the flow map, a time dependent family of orientation preserving diffeomorphisms $x(t, X)$, $0 \le t \le T$. The material points X (Lagrangian coordinate) in the reference configuration are deformed to the spatial position $x(t, X)$ (Eulerian cooedinate) at time t. Given a vector field $u(t, x)$, the flow map is determined by solving the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x(t, X) = u\big(t, x(t, X)\big), \\ x(0, X) = X. \end{cases} \tag{1.2}$$

Another important quantity governing the dynamics is the deformation tensor F:

$$F(t, X) = \frac{\partial x}{\partial X}(t, X). \tag{1.3}$$

The velocity field $u(t, x)$ determines the flow map, and hence the whole dynamics. However, in order to describe the changing of any configuration during such dynamical processes, or "elastic" properties of materials, the deformation tensor F is involved.

The action functional is taken of the form

$$\mathcal{L}(x) = \int_{\mathbf{R}^n} \Big(\frac{1}{2}|x_t(t, X)|^2 - h\big(\rho(t, X)\big)\Big)\mathrm{d}X, \tag{1.4}$$

where h is the strain energy function, and the density in the undeformed reference configuration has been set equal to one:

$$\rho = \frac{1}{\det F}. \tag{1.5}$$

Formal variation of the action functional $\mathcal{L}$ with respect to the flow map x gives the force balance equation

$$\rho(u_t + u\nabla \cdot u) + \nabla p = 0,$$

where $p(\rho) = \rho^2 h'(\rho)$, and we have used (1.2), (1.3), (1.5) and the identity

$$\frac{\partial \det F}{\partial F} = F^{-T} \det F.$$

For more details, see [4]–[7].

When viscosity is present, an additional diffusion term is present in the momentum equation. This gives the momentum equation in the equation (1.1). Finally, we get the following energy law:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{R}^n} \Big(\frac{1}{2}\rho|u|^2 + \rho h(\rho)\Big)\mathrm{d}x + \mu\|\nabla u\|^2_{L^2(\mathbf{R}^n)} + (\lambda + \mu)\|\nabla \cdot u\|^2_{L^2(\mathbf{R}^n)} = 0, \tag{1.6}$$

which governs the compressible Navier-Stokes equations (1.1).

We need impose the following constraint on the strain energy function:

$$h(\rho) \ge \frac{1}{2}|\rho - 1|^2, \tag{1.7}$$

which guarantees the positivity of the total energy. The initial data take the form

$$\rho(0, x) = 1 + \rho_0(x), \quad u(0, x) = u_0(x), \qquad x \in \mathbf{R}^3. \tag{1.8}$$

The main theorem of this paper can be stated as follows.