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A Uniaxial Optimal Perfectly Matched Layer Method for Time-harmonic Scattering Problems*

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Abstract: We develop a uniaxial optimal perfectly matched layer (opt PML) method for solving the time-harmonic scattering problems by choosing a particular absorbing function with unbounded integral in a rectangular domain. With this choice, the solution of the optimal PML problem not only converges exponentially to the solution of the original scatting problem, but also is insensitive to the thickness of the PML layer for sufficiently small parameter ε_0. Numerical experiments are included to illustrate the competitive behavior of the proposed optimal method.

Key words: uniaxial optimal perfectly matched layer, time-harmonic scattering, convergence

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1 Introduction

The first problem to be tackled for the numerical solution of any scattering problem in an unbounded domains is to truncate the computational domain without perturbing too much the solution of the original problem. Perfectly Matched Layer (PML) technique has been developed with this purpose which was introduced by Berenger[1]. This method has been applied to different problems (see [2]–[6]). The adaptive PML technique was proposed in [7] for a scattering problem by periodic structures (the grating problem), in [8] for acoustic scattering problem using circular PML domain, and in [9] for scattering problem using rectangular PML domain.

Bermudez and Hervella-Nieto[10] have studied an optimal bounded PML technique by choosing a particular absorbing function with unbound integral. They proved that such

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choice is easy to implement in a finite element method and overcomes the dependency of parameters for the discrete problem. But the PML equations in [10] are of singular coefficients and it is difficult to analyse the convergence of the PML solution.

In this paper we extend the idea of [9] and [10], and choose the PML parameter

$$\sigma_j(t) = \frac{1}{(L_j/2 + d_j + \varepsilon_0 - |t|)^m} - \frac{1}{(d_j + \varepsilon_0)^m},$$

where ε_0 is a sufficiently small positive constant, $m \geq 2$, L_1 and L_2 are respectively the length and width of computational domain

$$B_1 = \{x \in \mathbf{R}^2 : |x_1| < L_1/2,\ |x_2| < L_2/2\},$$

d_j is the thickness of PML layer, $j = 1, 2$. The singularity of coefficents of PML equations disappears. We prove that the PML solution $\tilde{u}$ converges exponentially to the solution of the original scattering problem u and the following estimation holds:

$$\|u - \tilde{u}\|_{H^1(\Omega_1)} \leq \hat{C}^{-1} C k^4 (1 + kL)^2 \varepsilon_0{}^{-5m} \mathrm{e}^{-\frac{\gamma k}{2(m-1)} \cdot \frac{1}{\varepsilon_0^{m-1}}} \|\tilde{u}\|_{H^{1/2}(\Gamma_1)},$$

where

$$\gamma = \frac{\min\{d_1, d_2\}}{\sqrt{(L_1 + d_1)^2 + (L_2 + d_2)^2}}, \quad m \geq 2, \quad L = \max\{L_1, L_2\},$$

$C > 0$ is independent of k, d_j, ε_0 and $\hat{C} \leq 1$. From this result we can see that the convergence is insensitive to the thickness of the PML layers for sufficiently small ε_0. So the total computational costs can be saved by choosing the thinner PML layer.

2 The Optimal PML Formulation

We study the following problem:

$$\begin{cases} \triangle u + k^2 u = 0\,, & \text{in } \mathbf{R}^2 \backslash \bar{D}; \\ \dfrac{\partial u}{\partial n} = -g, & \text{on } \Gamma_D; \\ \lim\limits_{r \to +\infty} \sqrt{r}\Big(\dfrac{\partial u}{\partial r} - \mathrm{i}ku\Big) = 0, & \end{cases} \tag{2.1}$$

where $D \subset \mathbf{R}^2$ is a bounded domain with Lipschitz boundary Γ_D, $g \in H^{-1/2}(\Gamma_D)$ is determined by the incoming wave, and n is the unit outer normal vector of Γ_D, $r = |x|$. We assume that $k \in \mathbf{R}$ is a constant. Let D be contained in the interior of the rectangular domain

$$B_1 = \{x \in \mathbf{R}^2 : |x_1| < L_1/2,\ |x_2| < L_2/2\}.$$

And let $\Gamma_1 = \partial B_1$, n_1 be the unit outer normal vector of Γ_1. For given $f \in H^{1/2}(\Gamma_1)$, similar to [9] we introduce the Dirichlet-to-Neumann operator

$$T : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1), \qquad Tf = \frac{\partial \xi}{\partial n_1}\Big|_{\Gamma_1},$$

where ξ is the solution of the following problem:

$$\begin{cases} \triangle \xi + k^2 \xi = 0\,, & \text{in } \mathbf{R}^2 \backslash \bar{B}_1; \\ \xi = f, & \text{on } \Gamma_1; \\ \lim\limits_{r \to +\infty} \sqrt{r}\Big(\dfrac{\partial \xi}{\partial r} - \mathrm{i}k\xi\Big) = 0. & \end{cases} \tag{2.2}$$