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Lie Higher Derivations on Nest Algebras*

Qi Xiao-fei[1] and Hou Jin-chuan[1,2]
(*1. School of Mathematical Sciences, Shanxi University, Taiyuan, 030006*)
(*2. Department of Mathematics, Taiyuan University of Technology, Taiyuan, 030024*)

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Abstract: Let $\mathcal{N}$ be a nest on a Banach space X, and Alg$\mathcal{N}$ be the associated nest algebra. It is shown that if there exists a non-trivial element in $\mathcal{N}$ which is complemented in X, then $D = (L_n)_{n\in\mathbf{N}}$ is a Lie higher derivation of Alg$\mathcal{N}$ if and only if each L_n has the form $L_n(A) = \tau_n(A) + h_n(A)I$ for all $A \in$ Alg$\mathcal{N}$, where $(\tau_n)_{n\in\mathbf{N}}$ is a higher derivation and $(h_n)_{n\in\mathbf{N}}$ is a sequence of additive functionals satisfying $h_n([A, B]) = 0$ for all $A, B \in$ Alg$\mathcal{N}$ and all $n \in \mathbf{N}$.
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1 Introduction

Let $\mathcal{A}$ be an associative algebra over a field $\mathbb{F}$. Then $\mathcal{A}$ is a Lie algebra under the Lie product

$$[A, B] = AB - BA.$$

Recall that an additive (linear) map δ from $\mathcal{A}$ into itself is called an additive (linear) derivation if

$$\delta(AB) = \delta(A)B + A\delta(B), \qquad A, B \in \mathcal{A}.$$

Since derivations are very important maps associated with homomorphisms, they are studied intensively (see [1] and [2]).

More generally, an additive (linear) map L from $\mathcal{A}$ into itself is called an additive (linear) Lie derivation if

$$L([A, B]) = [L(A), B] + [A, L(B)], \qquad A, B \in \mathcal{A}.$$

Lie derivations are a kind of additive (linear) maps that associated with Lie homomorphisms closely. Of course, all derivations are Lie derivations. The converse problem of whether an

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additive (linear) Lie derivation is an additive (linear) derivation has received many mathematicians' attention for many years. Brešar[3] proved that every additive Lie derivation on a prime ring $\mathcal{R}$ with characteristic not 2 can be decomposed as $\delta+\zeta$, where δ is a derivation from $\mathcal{R}$ into its central closure and ζ is an additive map of $\mathcal{R}$ into the extended centroid $\mathcal{C}$ sending commutators to zero. Mathieu and Villena[4] proved that every linear Lie derivation on a C^*-algebra can be decomposed into the sum of a derivation and a center-valued trace. Zhang[5] proved the same result for nest subalgebras of factor von Neumann algebras.

On the other hand, higher derivations were introduced and studied mainly in commutative rings, and later, also in non-commutative rings (see [6]–[8]). We first recall the concepts of higher derivations and Lie higher derivations.

Definition 1.1 *Let $D=(L_i)_{i\in\mathbf{N}}$ be a sequence of additive maps of a ring $\mathcal{R}$ such that $L_0=id_{\mathcal{R}}$. D is said to be:*

a higher derivation (HD, for short) if for every $n\in\mathbf{N}$ we have

$$L_n(AB)=\sum_{i+j=n}L_i(A)L_j(B),\qquad A,B\in\mathcal{R};$$

a Lie higher derivation (LHD, for short) if for every $n\in\mathbf{N}$ we have

$$L_n([A,B])=\sum_{i+j=n}[L_i(A),\ L_j(B)]\qquad A,B\in\mathcal{R}.$$

Higher derivations and Lie higher derivations are also related to homomorphisms and Lie homomorphisms in a natural way, respectively. Let $\mathcal{A}$ be a ring and

$$\mathcal{B}=\Big\{\sum_{n=0}^{\infty}A_n\lambda^n:\ A_n\in\mathcal{A}\Big\},$$

where the sum $\sum\limits_{n=0}^{\infty}A_n\lambda^n$ is formal. Let $D=(L_n)_{n\in\mathbf{N}}$ be a sequence of additive maps on $\mathcal{A}$ such that $L_0=id_{\mathcal{A}}$, and define a map $\Phi:\mathcal{A}\to\mathcal{B}$ by

$$\Phi(A)=\sum_{n=0}^{\infty}L_n(A)\lambda^n,\qquad A\in\mathcal{A}.$$

Then it is easily checked that $D=(L_n)_{n\in\mathbf{N}}$ is a higher derivation if and only if Φ is a homomorphism (see [9], Exercise 4, P.540]); $D=(L_n)_{n\in\mathbf{N}}$ is a Lie higher derivation if and only if Φ is a Lie homomorphism.

It is clear that all higher derivations are Lie higher derivations. However, the converse is not true in general. Assume that $D=(D_n)_{n\in\mathbf{N}}$ is a higher derivation on a ring $\mathcal{A}$. For any $n\in\mathbf{N}$, let

$$L_n=D_n+h_n,$$

where h_n is an additive map from $\mathcal{A}$ into its center vanishing on every commutator. It is easily seen that $(L_n)_{n\in\mathbf{N}}$ is a Lie higher derivation, but not a higher derivation if

$$h_n\neq 0\qquad\text{for some } n.$$

Then, a natural question is to ask whether or not every Lie higher derivation has the above form?