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A Sufficient Condition for the Genus of an Annulus Sum of Two 3-manifolds to Be Non-degenerate*

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Abstract: Let M_i be a compact orientable 3-manifold, and A_i a non-separating incompressible annulus on a component of ∂M_i, say F_i, $i = 1, 2$. Let $h : A_1 \to A_2$ be a homeomorphism, and $M = M_1 \cup_h M_2$, the annulus sum of M_1 and M_2 along A_1 and A_2. Suppose that M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with distance $d(S_i) \geq 2g(M_i) + 2g(F_{3-i}) + 1$, $i = 1, 2$. Then $g(M) = g(M_1) + g(M_2)$, and the minimal Heegaard splitting of M is unique, which is the natural Heegaard splitting of M induced from $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$.

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1 Introduction

Let M_i be a compact connected orientable bordered 3-manifold, and A_i an incompressible annulus on ∂M_i, $i = 1, 2$. Let $h : A_1 \to A_2$ be a homeomorphism. The manifold M obtained by gluing M_1 and M_2 along A_1 and A_2 via h is called an annulus sum of M_1 and M_2 along A_1 and A_2, and is denoted by $M_1 \cup_h M_2$ or $M_1 \cup_{A_1=A_2} M_2$.

Let $V_i \cup_{S_i} W_i$ be a Heegaard splitting of M_i for $i = 1, 2$, and

$$M = M_1 \cup_{A_1=A_2} M_2.$$

Then from Schultens[1], we know that M has a natural Heegaard splitting $V \cup_S W$ induced from $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ with genus

$$g(S) = g(S_1) + g(S_2).$$

So we always have

$$g(M) \leq g(M_1) + g(M_2).$$

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Let K_i be a knot in a closed 3-manifold N_i, $i = 1, 2$, and (N, K) the connected sum of pairs (N_1, K_1) and (N_2, K_2), i.e., $(N, K) = (N_1 \sharp N_2, K_1 \sharp K_2)$. Let $\eta(K)$ be an open regular neighborhood of K in N and the exterior $E(K) = N - \eta(K)$. Let A be the decomposing annulus in $E(K)$ which splits $E(K)$ into $E(K_1)$ and $E(K_2)$. Then

$$E(K) = E(K_1) \cup_{A_1=A_2} E(K_2),$$

where A_1 is a copy of A in $E(K_1)$, and A_2 is a copy of A in $E(K_2)$. Thus

$$g(E(K)) \leq g(E(K_1)) + g(E(K_2)).$$

Note that

$$g(E(K)) = t(K) + 1,$$

where $t(K)$ is the tunnel number of K, so

$$t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$$

always holds.

When $g(M) < g(M_1) + g(M_2)$, we say that the genus of the annulus sum is degenerate. Otherwise, it is non-degenerate. There exist examples which show that $g(M) < g(M_1) + g(M_2)$ could hold. For example, it has been shown in [2] and [3] that for any integer n, there exist infinitely many pairs of knots K_1, K_2 in S^3 such that

$$t(K_1 \# K_2) \leq t(K_1) + t(K_2) - n.$$

Note that for a knot K in S^3, $g(E(K)) = t(K) + 1$. So

$$g(E(K_1 \# K_2)) \leq g(E(K_1)) + g(E(K_2)) - n - 1.$$

In this paper, we give a sufficient condition for the genus of an annulus sum of two 3-manifolds to be non-degenerate in terms of distances of the factor Heegaard splittings.

The paper is organized as follows. In Section 2, we review some preliminaries which will be used later. The statement of the main result and its proof are included in Section 3. All 3-manifolds in this paper are assumed to be compact and orientable.

2 Preliminaries

In this section, we review some fundamental facts on surfaces in 3-manifolds. Definitions and terms which have not been defined are all standard; refer to, for examples, [4].

A Heegaard splitting of a 3-manifold M is a decomposition $M = V \cup_S W$ in which V and W are compression bodies such that

$$V \cap W = \partial_+ V = \partial_+ W = S$$

and

$$M = V \cup W.$$

S is called a Heegaard surface of M. The genus $g(S)$ of S is called the genus of the splitting $V \cup_S W$. We use $g(M)$ to denote the Heegaard genus of M, which is equal to the minimal genus of all Heegaard splittings of M. A Heegaard splitting $V \cup_S W$ for M is minimal if $g(S) = g(M)$. $V \cup_S W$ is said to be weakly reducible (see [5]) if there are essential disks $D_1 \subset V$ and $D_2 \subset W$ with $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, $V \cup_S W$ is strongly irreducible.