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L^p Harmonic k-forms on Complete Noncompact Hypersurfaces in $\mathbb{S}^{n+1}$ with Finite Total Curvature

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Abstract. In general, the space of L^p harmonic forms $\mathcal{H}^k(L^p(M))$ and reduced L^p cohomology $H^k(L^p(M))$ might be not isomorphic on a complete Riemannian manifold M, except for $p=2$. Nevertheless, one can consider whether $\dim\mathcal{H}^k(L^p(M))<+\infty$ are equivalent to $\dim H^k(L^p(M))<+\infty$. In order to study such kind of problems, this paper obtains that dimension of space of L^p harmonic forms on a hypersurface in unit sphere with finite total curvature is finite, which is also a generalization of the previous work by Zhu. The next step will be the investigation of dimension of the reduced L^p cohomology on such hypersurfaces.

AMS subject classifications: 53C21

Key words: L^p harmonic k-form, hypersurface in sphere, total curvature.

1 Introduction

Recall that on a complete Riemannian manifold M with dimension n, a differential form $\alpha\in\Omega^k(M)$, $(0\leq k\leq n)$ is called an L^p differential form if

$$\int_M |\alpha|^p \,\mathrm{d}v<+\infty, \text{ i.e., } |\alpha|\in L^p(M),$$

and we define

$$\Omega^k(L^p(M)):=\{\alpha\in\Omega^k(M)\mid |\alpha| \text{ and } |\mathrm{d}\alpha|\in L^p(M)\}.$$

In order to establish the corresponding de Rham cohomology, one considers the space of L^p closed forms

$$Z^k(L^p(M)):=\{\alpha\in\Omega^k(L^p(M))\mid \mathrm{d}\alpha=0\},$$

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where the equation $\mathrm{d}\alpha=0$ holds in weak sense, i.e.,

$$\int_M \langle \alpha,\delta\beta\rangle =0, \text{ for any } \beta\in\Omega_0^k(M),$$

and the space of L^p exact forms

$$B^k(L^p(M)):=\mathrm{d}\Omega^{k-1}(\mathrm{d}).$$

Then one can defined k-th de Rham space of unreduced L^p-cohomology as

$$H^k_{ur}(L^p(M)):=\frac{Z^k(L^p(M))}{B^k(L^p(M))},$$

and k-th de Rham space of reduced L^p-cohomology as

$$H^k(L^p(M)):=\frac{Z^k(L^p(M))}{\overline{B^k(L^p(M))}},$$

where $\overline{B^k(L^p(M))}$ is the closure of $B^k(L^p(M))$. For more information on L^p cohomology theory, please refer to [3,9,15] and references therein.

For compact manifolds, Hodge Theorem says that de Rham cohomology can be computed by harmonic forms. This motivates us to study L^p harmonic forms. Notice that on a compact Riemannian manifold a harmonic form α satisfies $\Delta\alpha=0$, where $\Delta:=-(\delta\mathrm{d}+\mathrm{d}\delta)$ is the Hodge Laplacian, and this is equivalent to $\mathrm{d}\alpha=0,\delta\alpha=0$. By the Gaffney cut-off trick, the same equivalence holds for L^2 harmonic forms on a complete Riemannian manifold [18], and $\mathcal{H}^k(L^2(M))$ is isomorphic to $H^k(L^2(M))$; however, in general, it is not the case for L^p $(p\neq 2)$. Actually, Alexandru-Rugina [1] found L^p integrable k-forms α satisfying $\Delta\alpha=0$, which are neither closed nor co-closed, on hyperbolic space $\mathbb{H}^n$ for $n\geq 3$. From this we see that the L^p $(p\neq 2)$ and L^2 harmonic theory are much different. In this paper, similar to [14, 3622], we define the space of L^p harmonic k-forms to be

$$\mathcal{H}^k(L^p(M)):=\{\alpha\in\Omega^k(L^p(M)) \mid \mathrm{d}\alpha=0,\delta\alpha=0\},$$

and define

$$\mathcal{H}_k(L^p(M)):=\ker(\Delta)\cap\Omega^k(L^p(M)).$$

Besides, Alexandru-Rugina [1] observed that for manifolds with bounded geometry, there is a continuous embedding

$$\mathcal{H}_k(L^p(M))\hookrightarrow H^k(L^p(M)).$$

Hence, if the dimension of $H^k(L^p(M))$ is finite, then so is the dimension of $\mathcal{H}_k(L^p(M))$. However, the opposite implication is unknown. Hence, one can ask the following question: