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Bifurcation of Equilibria in a Class of Planar Piecewise Smooth Systems with 3 Parameters*

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Abstract: The goal of this paper is to investigate the bifurcation properties of stationary points of a class of planar piecewise smooth systems with 3 parameters using the theory of differential inclusions. We especially study the existence of the stationary points on the line of discontinuity of this kind of planar piecewise smooth system.

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1 Introduction

Piecewise smooth systems are increasingly used to model a wide variety of physical systems and technological devices in engineering and applied science. These practical problems lead us to consider dynamical systems that are piecewise smooth. The examples of such systems include relay feedback systems in control theory, switching circuits in power electronics, mechanical systems with friction or impacts, a car brake system, etc. (see [1]–[4]).

Smooth dynamical systems and piecewise smooth systems both can possess stationary states or equilibria which can be stable or unstable. It is often necessary to know how the equilibria of a system change when a parameter of the system is varied. The number and stability of equilibria can change at a certain critical parameter value. And this qualitative change in the structural behavior of the system is called bifurcation. The theory of bifurcations of equilibria in smooth systems is well known (see [5] and [6]). However, much less is known about bifurcations of equilibria in piecewise smooth systems.

Piecewise smooth systems can exhibit bifurcations like the equilibrium-point bifurcations which can be seen in smooth systems. These bifurcations include saddle-node, transcritical, pitchfork and hopf-like bifurcations. However, some unique equilibrium-point bifurcations which can not be seen in smooth systems have been found in piecewise smooth systems

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because of their non-smoothness. For example, Leine[7] gave a detailed analysis of different types of non-smooth bifurcations scenarios for equilibrium points. These bifurcations are called "discontinuous bifurcations" by the author. For each of the classical smooth bifurcations, a "discontinuous bifurcation" was found to exist, and this result was illustrated with appropriate low-dimensional examples. Leine[8] showed a variety of bifurcation phenomena of equilibria that can be observed in non-smooth continuous systems. The so-called "multiple crossing bifurcations" can be found in non-smooth systems, for which the eigenvalues jump more than once over the imaginary axis, and which do not have a classical bifurcation as counterpart. Giannakopoulos and Pliete[9] investigated the bifurcation of equilibrium points and periodic trajectories of the $\mathbb{Z}_2$-symmetry planar piecewise linear differential equations. Zheng *et al.*[10] studied the bifurcation of equilibria of a planar piecewise smooth system when the discontinuity and the equilibria interact on each other, etc.

In this paper we investigate the bifurcation properties of stationary points of a class of planar piecewise linear system with 3 parameters of the form

$$\dot{u} = \begin{cases} A^+(u - \lambda_1 e_2 - \lambda_2 e_1), & x > 0; \\ A^-(u + \lambda_1 e_2 + \lambda_3 e_1), & x < 0, \end{cases} \tag{1.1}$$

where $u = (x, y)^T \in \mathbf{R}^2$, $A^\pm = \begin{pmatrix} a_{11}^\pm & a_{12}^\pm \\ a_{21}^\pm & a_{22}^\pm \end{pmatrix}$ is a 2×2 real matrix , e_1 and e_2 are natural bases of $\mathbf{R}^2$, $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbf{R}^3$ are parameters. The equation (1.1) is a piecewise linear Filippov system. According to [11], the isolated singular point $O(0,0)$ of system (1.1) with $\lambda = 0$ cannot have its degree of structural instability less than 3. The unfolding problem of system (1.1) lead us to consider the bifurcation property with at least 3 parameters. Giannakopoulos and Pliete[9] investigated the bifurcation of equilibrium points of the $\mathbb{Z}_2$-symmetry planar piecewise linear differential equations with two parameters. In this paper, we especially study the existence of the stationary points on the line of discontinuity of this planar piecewise smooth system.

2 Basic Assumptions

Consider the planar piecewise linear differential equation (1.1). We make some assumptions as follows.

(H1) $4\det(A^\pm) > (\mathrm{tr}(A^\pm))^2 \neq 0$.

(H2) $a_{12}^\pm > 0$.

(H3) $a_{12}^- a_{22}^+ - a_{12}^+ a_{22}^- = 0$.

From assumption (H1) it follows that the matrix $A^\pm$ possesses a pair of complex eigenvalues $\alpha^\pm \pm i\beta^\pm$ and $\alpha^\pm \neq 0$. Without losing generality we assume $\beta^\pm > 0$. Assumption (H2) assures that the flow of equation (1.1) crosses the y-axis clockwise. And it is obvious that the line of discontinuity of system (1.1) is

$$\Sigma = \{(x, y)^T \in \mathbf{R}^2 : x = 0\}.$$

Let

$$f^+(u, \lambda) = A^+(u - \lambda_1 e_2 - \lambda_2 e_1), \qquad f^-(u, \lambda) = A^-(u + \lambda_1 e_2 + \lambda_3 e_1).$$