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Estimates for Green's Functions of Elliptic Equations in Non-Divergence Form with Continuous Coefficients

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Abstract. We present a new method for the existence and pointwise estimates of a Green's function of non-divergence form elliptic operator with Dini mean oscillation coefficients. We also present a sharp comparison with the corresponding Green's function for constant coefficients equations.

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Key words: Green's function, Elliptic equations in nondivergence form, Dini mean oscillation coefficients.

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1 Introduction and main results

We consider a second-order elliptic operator L in non-divergence form

$$Lu = a^{ij}(x) D_{ij} u, \tag{1.1}$$

where the coefficient $\mathbf{A} := (a^{ij})$ are symmetric and satisfy the uniform ellipticity condition. Namely,

$$a^{ij} = a^{ji}, \quad \lambda |\xi|^2 \le a^{ij}(x) \xi^i \xi^j \le \Lambda |\xi|^2, \tag{1.2}$$

for some positive constants λ and Λ in a domain $\Omega \subset \mathbb{R}^n$ with $n \ge 3$. Here and below, we use the usual summation convention over repeated indices.

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In this article, we are concerned with construction and pointwise estimates for the Green's function $G(x,y)$ of the non-divergent operator L in Ω. In a recent article [15], it is shown that if the coefficients $\mathbf{A}$ is of Dini mean oscillation and the domain Ω is bounded and has $C^{2,\alpha}$ boundary, then the Green's function exists and satisfies the pointwise bound

$$|G(x,y)| \le C|x-y|^{2-n}. \tag{1.3}$$

Before proceeding further, let us introduce the definition of Dini mean oscillation. For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the Euclidean ball with radius r centered at x, and write $\Omega(x,r) := \Omega \cap B(x,r)$. We denote

$$\omega_{\mathbf{A}}(r,x) := \fint_{\Omega(x,r)} |\mathbf{A}(y) - \bar{\mathbf{A}}_{\Omega(x,r)}| dy, \quad \text{where} \quad \bar{\mathbf{A}}_{\Omega(x,r)} := \fint_{\Omega(x,r)} \mathbf{A},$$

and we write

$$\omega_{\mathbf{A}}(r,D) := \sup_{x\in D} \omega_{\mathbf{A}}(r,x) \quad \text{and} \quad \omega_{\mathbf{A}}(r) = \omega_{\mathbf{A}}(r,\bar{\Omega}). \tag{1.4}$$

We say that $\mathbf{A}$ is of Dini mean oscillation in Ω if $\omega_{\mathbf{A}}(r)$ satisfies the Dini condition; i.e.,

$$\int_0^1 \frac{\omega_{\mathbf{A}}(t)}{t} dt < +\infty.$$

It is clear that if $\mathbf{A}$ is Dini continuous, then $\mathbf{A}$ is of Dini mean oscillation. Also if $\mathbf{A}$ is of Dini mean oscillation, then $\mathbf{A}$ is uniformly continuous in Ω with its modulus of continuity controlled by $\omega_{\mathbf{A}}$; see [15, Appendix]. However, a function of Dini mean oscillation is not necessarily Dini continuous; see [7] for an example.

The main result of [15] is interesting because unlike the Green's function for uniformly elliptic operators in divergence form, the Green's function for non-divergent elliptic operators does not necessarily enjoy the pointwise bound (1.3) even in the case when the coefficient $\mathbf{A}$ is uniformly continuous; see [1]. It should be noted that the Dini mean oscillation condition is the weakest assumption in the literature that guarantees the pointwise bound (1.3). The proof in [15] is based on considering approximate Green's functions (as in [13, 14]) and showing that they satisfy specific estimates, as well as a local L^∞ estimate for solutions to the adjoint equation $L^*u=0$, which is shown in [7,8]. This L^∞ estimate is crucial for the pointwise bound (1.3) and it is where the Dini mean oscillation condition is strongly used; a mere continuity of $\mathbf{A}$ is not enough to produce such an estimate. We should recall that the adjoint operator L^* is given by

$$L^*u = D_{ij}(a^{ij}(x)u).$$