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Optimal Control Problem for a Reaction-Diffusion System of Three Populations

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Abstract. The work deals with an optimal control problem for a reaction-diffusion system comprising two competing populations, one of which is a prey for a third population. In order to maximise the total density of these populations, the existence and uniqueness of a positive strong solution of a controlled system are studied. After that, the techniques of minimal sequences is used in order to show the existence of an optimal solution. The first and second order optimality conditions are also constructed.

AMS subject classifications: 65M10, 78A48

Key words: Optimal control, reaction-diffusion, controlled system, optimality conditions.

1. Introduction

Optimal control problems attract more and more attention. Here, we study a control problem related to three populations reaction-diffusion systems with homogeneous Neumann boundary conditions — viz.

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - d_1 \Delta u = u\left(a - u - \alpha_1 v - \frac{w}{\beta_1 + u}\right) && \text{in } Q_T,\\
&\frac{\partial v}{\partial t} - d_2 \Delta v = v(b - v - \alpha_2 u) && \text{in } Q_T,\\
&\frac{\partial w}{\partial t} - d_3 \Delta w = w\left(c - \frac{w}{\beta_2 + u}\right) && \text{in } Q_T,\\
&\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0 && \text{on } \Sigma_T,\\
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x) && \text{in } \Omega,
\end{aligned}
$$

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where $Q_T = \Omega \times (0,T)$, $T > 0$ is a fixed time, $\Omega \subset \mathbb{R}^d$, $d \le 3$ a bounded domain with the boundary $\partial\Omega$ from the class $C^{2+\sigma}$, $\sigma > 0$, and n the outward unit normal vector on $\partial\Omega$. Besides, Δ is the Laplace operator, $d_1, d_2, d_3 > 0$ the diffusion coefficients, and $a, b, c, \alpha_1, \alpha_2, \beta_1$ and β_2 are positive constants. We also note that u and v stand for the densities of competing populations, and w represents the density of the predators preying on u. The functions u, v and w depend on the spatial position $x \in \Omega$ and time $t \in [0,T]$, and the Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0$$

are imposed on $\Sigma_T = \partial\Omega \times (0,T)$. The initial conditions for all three populations are assumed to be

$$u_0(x) > 0, \quad v_0(x) > 0, \quad w_0(x) > 0.$$

Partial differential equations are a key research topic in mathematics [12–14] and many of them are related to our work on optimal control problems. Thus optimal control issues for the Lotka-Volterra system of ordinary differential equations and for three-population system with diffusion are studied by Apreutesei in [1] and [3], respectively. Optimal control strategies for systems modelled by reaction-diffusion equations are discussed in [2, 5]. Apreutesei [4] also considered optimal control of a plant, predator and pest system. Ding *et al.* [9] investigated optimal control strategy of a nutrient-phytoplankton-zooplankton-fish system. Barthel *et al.* [6] discussed optimal boundary control of a system of reaction diffusion equations. Optimal control strategies for a model with general Holling type functional response is studied in [21]. It differs from the classical law and the growth of predator population gradually reduces. Xiang and Liu, [20] considered an inverse problem of SIS epidemic reaction-diffusion model. Optimal control strategies in population dynamics can be also found in [8, 10, 11, 17, 18, 22, 23].

This work is devoted to an optimal control problem for a reaction-diffusion system. Our goal is to maximise the total density of three populations. In contrast to the existing studies, which are mainly focused on ordinary differential equations, we also take diffusion into account. Besides, an additional control term h is introduced to two competing populations, so that the corresponding analysis is more involved. This affects competing populations u and v. For the sake of simplicity, we assume that this control is to enhance their densities as a stimulant, and the enhancement is proportional to the population densities. The control term h varies in the interval $[0,1]$, so that the admissible control set can be expressed as

$$\mathcal{U}_{ad} = \left\{ h \in L^2(Q_T) : 0 \le h(x,t) \le 1 \quad \text{a.e. on } Q_T \right\}.$$

Our goal is to maximise the weighted density of three populations in Q_T and the weighted density in Ω at a time T. Set $p = (u, v, w)$. Consider the cost function

$$\inf_{h \in \mathcal{U}_{ad}} J(p,h) = -\int_{Q_T} (\gamma_1 u + \gamma_2 v + \gamma_3 w)(x,t)\,dx\,dt - \int_{\Omega} (\delta_1 u + \delta_2 v + \delta_3 w)(x,T)\,dx, \tag{1.1}$$