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Tensor Bi-CR Methods for Solutions of High Order Tensor Equation Accompanied by Einstein Product

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Abstract. Tensors have a wide application in control systems, documents analysis, medical engineering, formulating an n-person noncooperative game and so on. It is the purpose of this paper to explore two efficient and novel algorithms for computing the solutions $\mathcal{X}$ and $\mathcal{Y}$ of the high order tensor equation $\mathcal{A}*_P\mathcal{X}*_Q\mathcal{B}+\mathcal{C}*_P\mathcal{Y}*_Q\mathcal{D}=\mathcal{H}$ with Einstein product. The algorithms are, respectively, based on the Hestenes-Stiefel (HS) and the Lanczos types of bi-conjugate residual (Bi-CR) algorithm. The theoretical results indicate that the algorithms terminate after finitely many iterations with any initial tensors. The resulting algorithms are easy to implement and simple to use. Finally, we present two numerical examples that confirm our analysis and illustrate the efficiency of the algorithms.

AMS subject classifications: 15A24, 65H10, 15A69, 65F10

Key words: Hestenes-Stiefel (HS) type of bi-conjugate residual (Bi-CR) algorithm, Lanczos type of bi-conjugate residual (Bi-CR) algorithm, high order tensor equation, Einstein product.

1. Introduction

We shall require some definitions and notation from tensors. For integer $P>0$, denote by $\mathbb{R}^{N_1\times\cdots\times N_P}$ the space of all real tensors of order P. A tensor $\mathcal{A}\in\mathbb{R}^{N_1\times\cdots\times N_P}$ is an array indexed by an integer tuple $(i_1,\ldots,i_P)$ in the range $1\leq i_j\leq N_j$ $(j=1,\ldots,P)$, that is, $\mathcal{A}=(a_{i_1\ldots i_P})_{1\leq i_j\leq N_j}$ $(j=1,\ldots,P)$. The Einstein product of tensors is defined by the operation $*_P$ via

$$(\mathcal{A}*_P\mathcal{B})_{i_1\ldots i_d j_1\ldots j_d}=\sum_{k_1,\ldots,k_d=1}^{L_1,\ldots,L_M} a_{i_1\ldots i_M k_1\ldots k_M} b_{k_1\ldots k_M j_1\ldots j_Q},$$

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where $\mathcal{A} \in \mathbb{R}^{M_1\times\cdots\times M_P\times L_1\times\cdots\times L_P}$ and $\mathcal{B} \in \mathbb{R}^{L_1\times\cdots\times L_P\times N_1\times\cdots\times N_Q}$ [6, 20, 38]. The Einstein product of tensors plays very important roles in many fields such as in theory of relativity and continuum mechanics [6, 20, 33]. There recently has been growing interest in the Einstein product of tensors [28, 39, 46, 64]. When $P = Q = 1$, the Einstein product reduces to the standard matrix multiplication. The transpose of the tensor $\mathcal{A} \in \mathbb{R}^{M_1\times\cdots\times M_P\times L_1\times\cdots\times L_P}$ is defined as

$$(\mathcal{A}^T)_{i_1\dots i_P j_1\dots j_P} = (\mathcal{A})_{j_1\dots j_P i_1\dots i_P}.$$

The trace of the tensor $\mathcal{A} \in \mathbb{R}^{M_1\times\cdots\times M_P\times L_1\times\cdots\times L_P}$ is

$$\operatorname{tr}(\mathcal{A}) = \sum_{i_1,\dots,i_P} a_{i_1\dots i_P i_1\dots i_P}.$$

The inner product of two tensors $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{M_1\times\cdots\times M_P\times L_1\times\cdots\times L_Q}$ is defined by

$$\langle \mathcal{A}, \mathcal{B}\rangle = \operatorname{tr}\left(\mathcal{B}^T *_P \mathcal{A}\right).$$

When $\langle \mathcal{A}, \mathcal{B}\rangle = 0$, we say that $\mathcal{A}$ and $\mathcal{B}$ are orthogonal. The Frobenius norm of $\mathcal{A} \in \mathbb{R}^{M_1\times\cdots\times M_P\times L_1\times\cdots\times L_P}$ is defined as

$$\|\mathcal{A}\| = \sqrt{\sum_{i_1,\dots,i_P,j_1,\dots,j_P} (a_{i_1\dots i_P j_1\dots j_P})^2} = \sqrt{\langle \mathcal{A}, \mathcal{A}\rangle}.$$

For any tensors $\mathcal{A}, \mathcal{B}, \mathcal{X} \in \mathbb{R}^{M_1\times\cdots\times M_P\times N_1\times\cdots\times N_Q}$, $\mathcal{C} \in \mathbb{R}^{M_1\times\cdots\times M_P\times L_1\times\cdots\times L_P}$, $\mathcal{Y} \in \mathbb{R}^{L_1\times\cdots\times L_P\times N_1\times\cdots\times N_Q}$ and any scalar $\lambda \in \mathbb{R}$, it can readily be verified that [55, 58]

$$\begin{aligned}
\langle \mathcal{A}, \mathcal{B}\rangle &= \operatorname{tr}\left(\mathcal{B}^T *_P \mathcal{A}\right) = \operatorname{tr}\left(\mathcal{A} *_Q \mathcal{B}^T\right) \\
&= \operatorname{tr}\left(\mathcal{B} *_Q \mathcal{A}^T\right) = \operatorname{tr}\left(\mathcal{A}^T *_P \mathcal{B}\right) = \langle \mathcal{B}, \mathcal{A}\rangle, && (1.1)\\
\langle \lambda\mathcal{A}, \mathcal{B}\rangle &= \lambda\langle \mathcal{A}, \mathcal{B}\rangle, && (1.2)\\
\langle \mathcal{X}, \mathcal{C} *_P \mathcal{Y}\rangle &= \left\langle \mathcal{C}^T *_P \mathcal{X}, \mathcal{Y}\right\rangle, && (1.3)\\
(\mathcal{C} *_P \mathcal{Y})^T &= \mathcal{Y}^T *_P \mathcal{C}^T. && (1.4)
\end{aligned}$$

Definition 1.1 ([58]). Define the transformation $\phi_{ML} : \mathbb{R}^{M_1\times\cdots\times M_P\times L_1\times\cdots\times L_P} \to \mathbb{R}^{M\times L}$ with $M = M_1M_2\dots M_P$, $L = L_1L_2\dots L_P$ and $\phi_{ML}(\mathcal{A}) = A$ defined component-wise as

$$(\mathcal{A})_{i_1\dots i_P j_1\dots j_P} \to A_{st},$$

where $\mathcal{A} \in \mathbb{R}^{M_1\times\cdots\times M_P\times L_1\times\cdots\times L_P}$, $A \in \mathbb{R}^{M\times L}$, $s = i_P + \sum_{k=1}^{P-1}((i_k-1)\prod_{r=k+1}^{P} M_r)$ and $t = j_P + \sum_{k=1}^{P-1}((j_k-1)\prod_{r=k+1}^{P} L_r)$.

Lemma 1.1 ([58]). *For* $\mathcal{A} \in \mathbb{R}^{M_1\times\cdots\times M_P\times L_1\times\cdots\times L_P}$, $\mathcal{X} \in \mathbb{R}^{L_1\times\cdots\times L_P\times N_1\times\cdots\times N_Q}$ *and* $\mathcal{C} \in \mathbb{R}^{M_1\times\cdots\times M_P\times N_1\times\cdots\times N_Q}$ *we have*

$$\mathcal{A} *_P \mathcal{X} = \mathcal{C} \Leftrightarrow \phi_{ML}(\mathcal{A})\phi_{LN}(\mathcal{X}) = \phi_{MN}(\mathcal{C}).$$