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Smallest Singular Value Based Newton-Like Methods for Solving Quadratic Inverse Eigenvalue Problem

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Abstract. A Newton method for solving quadratic inverse eigenvalue problems is proposed. The method is based on the properties of the smallest singular value of a matrix. In order to reduce computational cost, we use approximations of the smallest singular value and the corresponding unit left and right singular vectors obtained by the one-step inverse iteration. It is shown that both the proposed method and its modification have locally quadratic convergence. Numerical results confirm theoretical findings and demonstrate the effectiveness of the methods proposed.

AMS subject classifications: 15A29, 65F18

Key words: Quadratic inverse eigenvalue problem, Newton method, Newton-like method, singular value, singular vector.

1. Introduction

Let M be a real $n \times n$ nonsingular matrix, $C(c)$ and $K(c)$ be affine families

$$C(c) = C_0 + \sum_{i=1}^{2n} c_i C_i, \quad K(c) = K_0 + \sum_{i=1}^{2n} c_i K_i, \tag{1.1}$$

where $\{C_i\}_{i=0}^{2n}, \{K_i\}_{i=0}^{2n} \in \mathbf{R}^{n\times n}$, $c = (c_1, \ldots, c_{2n})^T \in \mathbf{R}^{2n}$. The quadratic inverse eigenvalue problem (QIEP) under consideration is as follows [21, 22].

Problem 1.1 (QIEP). *For given distinct complex numbers $\{\lambda_i\}_{i=1}^{2n}$ (closed under complex conjugation), a nonsingular matrix $M \in \mathbf{R}^{n\times n}$ and matrices $\{C_i\}_{i=0}^{2n}, \{K_i\}_{i=0}^{2n} \in \mathbf{R}^{n\times n}$, find a vector $c \in \mathbf{R}^{2n}$ such that the quadratic eigenvalue problem $Q(\lambda, c)x = 0$ has the prescribed eigenvalues $\{\lambda_i\}_{i=1}^{2n}$, where*

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$$Q(\lambda, c) = \lambda^2 M + \lambda C(c) + K(c), \tag{1.2}$$

in which $C(c)$ *and* $K(c)$ *are defined by* (1.1).

QIEP plays an important role in various practical applications such as dynamic behavior control of damped mass-spring systems [22], structural design [25], finite element model updating [11,12] and so on. A special case of QIEP — viz. the inverse eigenvalue problem (IEP) when $M = 0$ and $C(c)$ is the identity matrix I, was studied in [7–10,24,29,34,37,38]. Another special case of QIEP, the generalized inverse eigenvalue problem (GIEP) — i.e. if $M = 0$, was investigated in [1,13,15,16,18–20,25,30]. Many efforts have been spent on the solvability of IEP and GIEP [15,38]. There are locally quadratic convergent methods for solving both IEP and GIEP, including Newton methods based on eigenvalues [8,9,16,24,25], Newton methods [7,24] based on determinant evaluations [27], Newton methods [13,15, 29] based on smooth QR/LU decompositions with column pivoting [14, 17, 28] and so on. Recently, based on the theory of multiparameter eigenvalue problems [3], Xiang and Dai [35] presented a sufficient condition for the solvability of QIEP. In this paper, we focus our attention on the numerical solution issue of QIEP.

There are several quadratically convergent methods for solving QIEP. Applying the Newton method based on determinant evaluations [7,24,27] to the system of nonlinear equations

$$d(c) = \begin{pmatrix} \det\left(\lambda_1^2 M + \lambda_1 C(c) + K(c)\right) \\ \det\left(\lambda_2^2 M + \lambda_2 C(c) + K(c)\right) \\ \vdots \\ \det\left(\lambda_{2n}^2 M + \lambda_{2n} C(c) + K(c)\right) \end{pmatrix} = 0, \tag{1.3}$$

Elhay and Ram [22] presented a Newton method for solving QIEP. However, this method may suffer from ill-conditioning [24] and is not computationally attractive. Assume that there exists a solution c^* to QIEP and $\{\lambda_i(c)\}_{i=1}^{2n}$ are the eigenvalues of the quadratic pencil $Q(\lambda, c)$. Then there is a neighborhood of c^* where the eigenvalues $\{\lambda_i(c)\}_{i=1}^{2n}$ of $Q(\lambda, c)$ are distinct differentiable functions [2]. The most natural formulation for solving QIEP consists in finding a root of the following system of nonlinear equations:

$$f(c) = \begin{pmatrix} \lambda_1(c) - \lambda_1 \\ \lambda_2(c) - \lambda_2 \\ \vdots \\ \lambda_{2n}(c) - \lambda_{2n} \end{pmatrix} = 0. \tag{1.4}$$

The Newton method based on the formulation (1.4) proposed by Bohte [8], was developed by Friedland *et al.* [24] for symmetric inverse eigenvalue problems, and further extended to symmetric generalized inverse eigenvalue problems by Dai and Lancaster [16]. Elhay and Ram [23] developed a Newton method for solving symmetric quadratic inverse eigenvalue problem under the assumption that the number of real and complex eigenvalues in each iteration remains the same as the number of real and complex eigenvalues prescribed. In order to keep the effectiveness of the Newton method, it is important to reorder the eigenvalues in a suitable way. However, even for symmetric matrices M, $C(c)$ and $K(c)$, the