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Preconditioned CG Methods for a Variable-Coefficient Nonlocal Diffusion Model

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Abstract. A variable-coefficient nonlocal diffusion model is discretized by an improved fast collocation scheme. The resulting linear system has a symmetric positive definite Toeplitz-like coefficient matrix. The preconditioned CG methods with Toeplitz and circulant preconditioners are used for solving the discretized linear system. Numerical experiments demonstrate the effectiveness of the preconditioned CG methods.

AMS subject classifications: 65F10, 65F15

Key words: Nonlocal diffusion model, fast collocation method, Toeplitz matrix, CG method, preconditioner.

1. Introduction

Nonlocal models represented by fractional differential equations [21, 27] provide an alternative tool for modeling of challenging phenomena such as anomalous diffusion and long-range spatial interactions, which cannot be modeled properly by conventional integer-order differential equations [16]. However, the numerical discretizations of such equations lead to dense stiffness matrices, so that the usual direct solvers require $\mathscr{O}(N^2)$ memory storage and $\mathscr{O}(N^3)$ computational cost, where N is the number of unknowns. The significantly increased computational complexity and the memory requirements become one of the main obstacles that hinders its applications.

In this paper, we consider a volume constrained Dirichlet boundary-value problem for a nonlocal diffusion model with a variable-coefficient [15, 16, 18, 20], viz.

$$\begin{aligned}&\int_{x-\delta}^{x+\delta}\big(\alpha(x)+\alpha(y)\big)\sigma(x-y)\big(u(x)-u(y)\big)dy=f(x),\quad x\in(a,b),\\&u(x)=g(x),\quad x\in(a-\delta,a]\cup[b,b+\delta).\end{aligned}\tag{1.1}$$

Here $\delta > 0$ refers to the size of the horizon specifying the range of the nonlocal diffu-

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sion phenomenon, and α is a smooth diffusivity coefficient with positive lower and upper bounds. The kernel $\sigma(x)$ has the form

$$\sigma(x) = \frac{1}{|x|^{1+\gamma}}. \tag{1.2}$$

The index $\gamma < 1$ specifies the intensity of the kernel, so the integral operator is weakly singular.

In the model (1.1), the variable coefficient occurs inside of the integral operator. This has a global impact on the discretization of the model, so that the number of summands in the matrix decomposition in [29, 30] will be of order $\mathcal{O}(N)$. Consequently, the corresponding matrix-vector multiplication is of order $\mathcal{O}(N^2)$. Recently, Che Wang and Hong Wang [28] developed an alternative approach to handle the impact of variable coefficients on the discretization of the nonlocal diffusion model. This leads to an optimal-order memory storage and approximately linear computational complexity of the numerical method. Although the storage and calculation of this numerical method are relatively small, the corresponding stiffness matrix is not symmetric. It is known that solving non-symmetric linear systems is a difficult problem, and to solve such system the conjugate gradient squared (CGS) method [31] and the fast CGS (FCGS) method have been used in [28]. In actual calculations, these two methods usually work well but because of rounding errors, the residual modulus often vibrates too much, causing computational instability and even overflow.

For this reason, we make a small change to the numerical scheme [28], so that the new stiffness matrix is symmetric positive definite under the premise that the storage amount and calculation amount remain the same. The change we made was to re-evaluate the elements of the stiffness matrix, so that we can apply the conjugate gradients (CG) method to the discretized linear system. The CG algorithm is one of the best known iterative techniques for solving sparse symmetric positive definite linear systems. The convergence of this method is related to the spectral distribution of the matrix. It works well if the matrix is well conditioned or if it has only a few distinct eigenvalues. If the spectrum of the matrix is more evenly distributed over a long interval, the CG method converges very slowly. Therefore, when the CG method is actually used, the corresponding linear system has to be preconditioned so that the spectrum of the coefficient matrix is relatively concentrated. This is the so-called preconditioned CG (PCG) method [17]. The use of a good preconditioner can have significant effect upon improving the convergence rate of the CG method — cf. [11] for more details.

Iterative solvers for Toeplitz systems are theoretically and numerically studied with numerous applications for over twenty years [13, 14, 22, 23]. However, the resulting discretized system for the variable-coefficient nonlocal diffusion model (1.1) is only a Toeplitz-like one. Therefore, the corresponding methods for Toeplitz systems do not work well in this situation. In recent years, several types of iteration methods have been developed for solving Toeplitz-like systems arising in the discretization of fractional partial differential equations [6, 7, 9, 10, 25, 26], linear third-order ordinary differential equations [2, 3, 12], and linear and nonlinear partial differential equations [1,4,5,8,24]. Motivated by the ideas of [19], we construct Toeplitz and circulant preconditioners for the Toeplitz-like linear sys-