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Complete and Complete Moment Convergence of the Weighted Sums of ρ^*-Mixing Random Vectors in Hilbert Spaces

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Abstract. Let $1 \le p < 2$, $\alpha > p$, $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be a set of real numbers with the property $\sup_{n\ge1} n^{-1}\sum_{i=1}^n |a_{ni}|^\alpha < \infty$ and let $\{X, X_n, n \ge 1\}$ be a sequence of H-valued ρ^*-mixing random vectors coordinatewise stochastically upper dominated by a random vector X. We provide conditions such that for any $\epsilon > 0$ the following inequalities hold:

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} a_{ni}X_i\right\| > \epsilon n^{\frac{1}{p}}\right) < \infty,$$

$$\sum_{n=1}^{\infty} n^{-1-\frac{1}{p}} E\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} a_{ni}X_i\right\| - \epsilon n^{\frac{1}{p}}\right)^{+} < \infty.$$

These results generalize the results of Chen and Sung (cf. J. Ineq. Appl. **121**, 1–16 (2018)) to the ρ^*-mixing random vectors in H. In addition, a Marcinkiewicz-Zygmund type strong law of ρ^*-mixing random vectors in H is presented.

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1. Introduction

We recall the concept of ρ^*-mixing random variables. Let $\{X_n, n \ge 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. For any subset $S \subset N = \{1, 2, \ldots\}$, define $\mathscr{F}_s = \sigma(X_i, i \in S)$. Given two σ-algebras $\mathscr{A}, \mathscr{B} \in \mathscr{F}$, set

$$\rho(\mathscr{A}, \mathscr{B}) := \sup\left\{\frac{EXY - EXEY}{\sqrt{E(X-EX)^2E(Y-EY)^2}} : X \in L_2(\mathscr{A}), Y \in L_2(\mathscr{B})\right\}.$$

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Define the ρ^*-mixing coefficient by

$$\rho_n^* := \sup\left\{\rho(\mathscr{F}_S, \mathscr{F}_T) : S, T \subset N \text{ with } \operatorname{dist}(S,T) \geq n\right\},$$

where

$$\operatorname{dist}(S,T) = \inf\left\{|s-t| : s \in S, t \in T\right\}.$$

Obviously, $0 \leq \rho_{n+1}^* \leq \rho_n^* \leq \cdots \leq \rho_0^* = 1$. A sequence $\{X_n, n \geq 1\}$ of random variables is called ρ^*-mixing if there exists $k \in N$ such that $\rho_k^* < 1$.

The limits of the weighted sums of random variables have been studied by Bai *et al.* [1], Chen *et al.* [2], Chen *et al.* [3], Chow [5], Cuzick [6], and Thrum [12].

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ generated by an inner product $<\cdot,\cdot>$, $\{e_j, j \geq 1\}$ an orthonormal basis in H, and X an H-valued random vector. In what follows, we write $X^{(j)}$ for $< X, e_j >$. Similar to the H-valued negatively associated random vectors — cf. [11], we consider the H-valued ρ^*-mixing random vectors.

Definition 1.1. *A sequence $\{X_n, n \geq 1\}$ of H-valued random vectors is said to be ρ^*-mixing if for any $d \geq 1$ the sequence $\{(X_n^{(1)}, X_n^{(2)}, \cdots, X_n^{(d)}), n \geq 1\}$ of $\mathbb{R}^d$-valued random vectors is ρ^*-mixing.*

In addition, similar to the notion of coordinatewise negatively associated H-valued random vectors [9], we consider coordinatewise ρ^*-mixing H-valued random vectors, which are more general than H-valued ρ^*-mixing random vectors.

Definition 1.2. *A sequence $\{X_n, n \geq 1\}$ of H-valued ρ^*-mixing random vectors is said to be coordinatewise ρ^*-mixing if for each $j \geq 1$, the sequence $\{X_n^{(j)}, n \geq 1\}$, $X_n^{(j)} =< X_n, e_j >$, $n \geq 1$ of random variables is ρ^*-mixing.*

It is clear that any ρ^*-mixing sequence of H-valued random vectors is coordinatewise ρ^*-mixing, but the reverse is not true in general.

Let us also recall the notion of complete convergence and complete moment convergence. The concept of complete convergence was introduced by Hsu and Robbins [7].

Definition 1.3. *A sequence $\{X_n, n \geq 1\}$ of random variables is said to converge completely to a constant c if for all $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} P\big(|X_n - c| > \epsilon\big) < \infty.$$

Chow [5] studied the complete moment convergence

Definition 1.4. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$. If for all $\epsilon > 0$*

$$\sum_{n=1}^{\infty} a_n E\left\{b_n^{-1}|X_n| - \epsilon\right\}^+ < \infty,$$

then the above result is said to be complete moment convergence.