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THE SCOTT-VOGELIUS METHOD FOR THE STOKES PROBLEM ON ANISOTROPIC MESHES

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Abstract. This paper analyzes the Scott-Vogelius divergence-free element pair on anisotropic meshes. We explore the behavior of the inf-sup stability constant with respect to the aspect ratio on meshes generated with a standard barycenter mesh refinement strategy, as well as a newly introduced incenter refinement strategy. Numerical experiments are presented which support the theoretical results.

Key words. Divergence free finite elements, anisotropic meshes, inf-sup stability.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a regular open polygon with boundary Γ. We consider the Stokes equation with the no-slip boundary condition:

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \text{ in } \Omega, \\ \boldsymbol{u} &= 0 \text{ on } \Gamma, \end{aligned}$$

where $\boldsymbol{u}$ is the velocity, p is the pressure, $\boldsymbol{f}$ is a given body force, and ν is the viscosity.

In this manuscript we study the stability of the divergence-free Scott-Vogelius (SV) finite element pair on anisotropic meshes for the Stokes problem; the results trivially extend to other divergence free equations, e.g., the incompressible Navier-Stokes equations. Divergence-free methods and other pressure-robust schemes are an extremely active field of research (cf. [25, 32]) ranging from a variety of finite element pairs (e.g., [35, 9, 37, 17, 1, 15, 22, 19, 21]) to modifying the formulation of the equations (e.g., [28, 14, 29, 30, 27, 36, 26]). Advantages of divergence-free methods include exact enforcement of conservation laws, pressure robustness with the velocity error being independent of the pressure error and viscosity term [25, 3], and improved stability and accuracy of timestepping schemes [11, 18].

The Stokes equation has been studied on anisotropic meshes for a number of different element pairs. In [8] it was shown that for the Crouzeix-Raviart element, the inf-sup constant is independent of the aspect ratio on triangular and tetrahedral meshes. A similar result was shown for the Bernardi-Raugel finite element pair in two dimensions for classes of triangular and quadrilateral meshes in [7]. Recently, in [10] it was shown for a specific class of anisotropic triangulation that the lowest order Taylor-Hood element was uniformly inf-sup stable. A nonconforming pressure robust method was studied in [6]. Stability and convergence on anisotropic meshes for the Stokes equation has also been studied extensively for the hp-finite element method [4, 33, 34].

Up to this point there have been no theoretical results for H^1 conforming divergence free finite elements on anisotropic meshes. The low-order SV element pair is

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somewhat unique in that it is not inf-sup stable on general meshes, but requires special meshes e.g., the barycenter refinement (or Clough-Tocher refinement) which is obtained by connecting the vertices of each triangle on a given mesh to its barycenter. As pointed out in [24, p.12] this gives rise to meshes with possibly very small and large angles. The impact of these angles on the inf-sup constant was stated as an open problem in [24].

In this work we show barycenter refinement on anisotropic meshes will necessarily lead to large angles and propose an alternative mesh refinement strategy based on the incenter of each triangle. This incenter refinement strategy produces a mesh that avoids large angles and allows a smaller increase in aspect ratio on refinement. We prove there is a linear relationship between the inf-sup constant and the inverse of the aspect ratio for both the barycenter and incenter refined mesh; numerical experiments show that these results are sharp. Surprisingly, numerical tests indicate that there is not a significant difference, in terms of accuracy, between the incenter and barycenter refinement.

The rest of this manuscript is organized as follows: In Section 2 we introduce notation and give some preliminary results that will be used for the inf-sup stability estimates. We also prove that the incenter refined mesh has superior aspect ratios and angles compared to the barycenter refined mesh. In Section 3 we prove that the inf-sup constant scales linearly with the inverse of the aspect ratio for both barycenter and incenter refinement. In Section 4, we verify numerically the geometric results proven in Section 2 and stability results proven in Section 3. We also demonstrate that there does not appear to be an appreciable difference in terms of accuracy for the incenter versus barycenter refinement. Finally, the appendix contains proofs of some technical lemmas.

2. Preliminaries

Let $\mathcal{T}_h$ denote a conforming simplicial triangulation of $\Omega \subset \mathbb{R}^2$. We denote the vertices and edges of T as $\{z_i\}_{i=1}^3$ and $\{e_i\}_{i=1}^3$ respectively, labeled such that z_i is opposite of e_i. Set $h_i = |e_i|$ and without loss of generality, we assume $h_1 \le h_2 \le h_3$. We denote by ρ_T the diameter of the incircle of T and set $h_T = h_3$. Let α_i be the angle of T at vertex z_i, note that $\alpha_1 \le \alpha_2 \le \alpha_3$.

Let $z_0 \in T$ be an interior point of T, and set $T^{ct} = \{K_1, K_2, K_3\}$ to be the local (Clough-Tocher) triangulation of T, obtained by connecting the vertices of T to z_0. The three triangles $\{K_i\}_{i=1}^3$ are labeled such that $\partial K_i \cap \partial T = e_i$. Let a_T be the altitude of T with respect to edge e_3, and let k_i be the altitude of K_i with respect to e_i (cf. Figure 1).

2.1. Geometric results and dependence of split point.

We examine the dependencies and properties of the local triangulation of T on the choice of split point z_0. In particular, we consider geometric properties of the triangulations obtained by connecting vertices of T to the barycenter and the incenter of T. First, we require a few definitions.

Definition 2.1. *The* barycenter *of T is given by*

$$z_{\text{bary}} = \frac{1}{3}(z_1 + z_2 + z_3).$$

The incenter *of T is given by*

$$z_{\text{inc}} = \frac{1}{|\partial T|}(h_1 z_1 + h_2 z_2 + h_3 z_3).$$