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ON AN ADAPTIVE LDG FOR THE *P*-LAPLACE PROBLEM

DONGJIE LIU, LE ZHOU*, AND XIAOPING ZHANG

Abstract. In this paper we consider the adaptive local discontinuous Galerkin(LDG) method for the p-Laplace problem in polygonal regions in $\mathbb{R}^2$. We present new sharper a posteriori error estimate for the LDG approximation of the p-Laplacian in the new framework. Several examples are given to confirm the reliability of the estimate.

Key words. p-Laplace, local discontinuous Galerkin methods, quasi-norm, a posteriori error estimate.

1. Introduction

Let Ω be a bounded polyhedral domain in $\mathbb{R}^2$ with polygonal boundary Γ. We consider the classical p-Laplace problem

$$(1)\qquad \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega \\ u = g_D = 0 & \text{on } \Gamma \end{cases},$$

for $2 < p < \infty$ and given $f \in L^q(\Omega)$ (q conjugate of p). The p-Laplace problem (1) admits a unique weak solution satisfying [7, 17]

$$(2)\qquad u = \arg\min E(v) \quad \text{for } v \in W_0^{1,p}(\Omega) := \{v \in W^{1,p}(\Omega), v|_\Gamma = 0\}.$$

where

$$(3)\qquad E(v) := \int_\Omega W(\nabla v)dx - \int_\Omega fv \ \mathrm{d}x.$$

The energy density function $W : \mathbb{R}^2 \to \mathbb{R}$ reads $W(\mathbf{a}) := |\mathbf{a}|^p/p$ with the derivative $\mathcal{A}(\mathbf{a}) := |\mathbf{a}|^{p-2}\mathbf{a}$ for all $\mathbf{a} \in \mathbb{R}^2$.

The Euler-Lagrange equation of (2) consists in finding $u \in W_0^{1,p}(\Omega)$ with

$$(4)\qquad \int_\Omega \mathcal{A}(\nabla u)\cdot\nabla v \ \mathrm{d}x - \int_\Omega fv \ \mathrm{d}x = 0 \quad \forall\ v \in W_0^{1,p}(\Omega).$$

The embedding of $W_0^{1,p}(\Omega)$ into $W_0^{1,2}$ is continuous when $2 < p < \infty$ and Ω is bounded domain (see [5]).

The p-Laplacian occurs in many mathematical models of physical processes such as glaciology, nonlinear diffusion and filtration, power-law materials, and quasi-Newtonian flows. Furthermore it is viewed as one of the typical examples of a large class of difficult problems-degenerate nonlinear systems.

The numerical approximation for p-Laplace problem has been studied extensively in the literature. The previous analysis of finite element method (FEM) for this kind of problem was undertaken in [12], where the error estimates have been shown in the $W^{1,p}$-norm. The results were further improved in [1, 13, 28]. Recently, sharper error estimates were derived in [2, 4, 20, 23, 24, 26] by developing the quasi-norm techniques.

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*Corresponding author.

Over the last two decades, there has been an increasing interest in discontinuous Galerkin (DG) methods for p-Laplace problem; see[6, 7, 17, 22]. Partically, Local discontinuous Galerkin (LDG) method [11, 14, 15] for p-structure problem was studied in [17, 22], where the quasi-norm interpolation estimates [18] were applied in the frame of broken spaces.

This paper aims at deriving a new explicit and reliable a posteriori error estimate for the LDG applied to the p-Laplacian. We generalize the Helmholtz decomposition of the gradient of the error [3, 9], derive the reliable a posteriori error estimate in the new defined distance, and the error of the energy can be presented at the same time in an easy way.

The remaining parts of this paper are organized as follows. In Section 2 we describe the LDG formulation and the equivalent minimization problem. In Section 3, we introduce the distance $\|F(\nabla u) - F(\nabla v)\|^2_{2,p,\Omega}$ to quantify the quality of approximations via $F(\mathbf{a}) := |\mathbf{a}|^{p/2-1}\mathbf{a}, \mathbf{a} \in \mathbb{R}^2$. The a posteriori error estimators based on new defined distance is presented in Section 4, Some numerical experiments conclude the paper in Section 5 with empirical evidence of the expected convergence.

Standard notation applies throughout this paper to Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$. Denote $\|\cdot\|_{L^p(\Omega)} := \|\cdot\|_{p,\Omega}$, $\|\cdot\|_{L^p(\Gamma)} := \|\cdot\|_{p,\Gamma}$. Denote the expression "$\lesssim$" abbreviates an inequality up to some multiplicative generic constant, i.e. $A \lesssim B$ means $A \le C\ B$ with some generic constant $0 \le C \le \infty$, which depends on the interior angles of the triangles but not their sizes. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2. Discontinuous finite element approximation

2.1. Discontinuous finite element space and Local L^2-projection. In order to obtain LDG formulation of (1), we introduce the gradient $\boldsymbol{\theta} := \nabla u$ and the flux $\boldsymbol{\sigma} := \mathcal{A}(\boldsymbol{\theta}) = |\nabla u|^{p-2}\nabla u$, then (1) can be reformulated as the follow problem: Find $(u, \boldsymbol{\theta}, \boldsymbol{\sigma})$ in appropriate space such that

$$(5) \qquad \begin{cases} \boldsymbol{\theta} = \nabla u, \boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\theta}), -\mathrm{div}\boldsymbol{\sigma} = f & \text{in } \Omega \\ u = g_D = 0 & \text{on } \Gamma \end{cases}.$$

Let $\mathcal{T}_h = \bigcup\{T\}$ be a shape-regular triangulation of $\bar{\Omega}$ such that $\bar{\Omega} = \bigcup\{T : T \in \mathcal{T}_h\}$, where straight triangle T has diameter h_T and unit outward normal to ∂T given by $\mathbf{n}_T$. $h := \max\{h_T : T \in \mathcal{T}_h\}$. We denote by $\Gamma_h = \bigcup\{E \subset \partial T : T \in \mathcal{T}_h\}$ the union of all edges of $\mathcal{T}_h$ and $\Gamma_I = \Gamma_h \backslash \Gamma$ an union of all interior edges of $\mathcal{T}_h$. The discontinuous finite element space of scalar function and vector function space are defined by

$$V_h = \{v \in L^p(\Omega) : v|_T \in \mathbf{P}_1(T) \quad \forall\, T \in \mathcal{T}_h\},$$
$$\Sigma_h = \{\boldsymbol{\theta} \in [L^p(\Omega)]^2 : \boldsymbol{\theta}|_T \in [\mathbf{P}_0(T)]^2 \quad \forall\, T \in \mathcal{T}_h\}.$$

$\mathbf{P}_k(T)$ denotes the polynomial of degree at most k on T. Similarly we have the piecewise smooth function space on $\mathcal{T}_h$

$$W^{1,p}(\mathcal{T}_h) = \{v \in L^p(\Omega) : v|_T \in W^{1,p}(T) \quad \forall\, T \in \mathcal{T}_h\}.$$

Let T_1 and T_2 be two adjacent elements with a common edge E. Denote $v_i := v|_{\partial T_i}$ the trace of function v restricted to E in element T_i with $\mathbf{n}_i := \mathbf{n}|_{\partial T_i}$ on E pointing exterior to T_i. Define jump and average of function v on E,

$$[\![v]\!] = v_1\mathbf{n}_1 + v_2\mathbf{n}_2, \quad \{v\} = \frac{1}{2}(v_1 + v_2), \quad E \in \Gamma_I.$$