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ON THE BOUNDEDNESS OF A CLASS OF NONLINEAR DYNAMIC EQUATIONS OF THE THIRD ORDER[*]

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Abstract

In this paper, a modified nonlinear dynamic inequality on time scales is used to study the boundedness of a class of nonlinear third-order dynamic equations on time scales. These theorems contain as special cases results for dynamic differential equations, difference equations and q-difference equations.

Keywords time scales; dynamic equation; integral inequality; boundedness; third-order

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1 Introduction

To unify and extend continuous and discrete analyses, the theory of time scales was introduced by Hilger [1] in his Ph.D.Thesis in 1988. Since then, the theory has been evolving, and it has been applied to various fields of mathematics; for example, see [2,3] and the references therein. It is well known that Gronwall-type integral inequalities and their discrete analogues play a dominant role in the study of quantitative properties of solutions of differential, integral and difference equations.

During the last few years, some Gronwall-type integral inequalities on time scales and their applications have been investigated by many authors. For example, we refer readers to [5-11]. In this paper, motivated by the paper [4], we obtain the bounds of the solutions of a class of nonlinear dynamic equations of the third order on time scales, which generalizes the main result of [4]. For all the detailed definitions, notation and theorems on time scales, we refer the readers to the excellent monographs [2,3] and references given therein. We also present some preliminary

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results that are needed in the remainder of this paper as useful lemmas for the discussion of our proof.

In what follows, R denotes the set of real number, $R_+ = [0, +\infty)$; $C(M, S)$ denotes the class of all continuous functions defined on a set M with range in a set S; T is an arbitrary time scale and C_{rd} denotes the set of rd-continuous functions. Throughout this paper, we always assume that $t_0 \in T$, $T_0 = [t_0, +\infty) \cap T$.

2 Preliminary

Lemma 2.1 *Suppose $u(t), a(t) \in C_{rd}(T_0, R_+)$, and a is nondecreasing, $f(t,s)$, $f_t^{\Delta}(t,s) \in C_{rd}(T_0 \times T_0, R_+)$, $\omega \in C(R_+, R_+)$ is nondecreasing. If for $t \in T_0$, $u(t)$ satisfies the following inequality*

$$u(t) \le a(t) + \int_{t_0}^{t} f(t,s)\omega(u(s))\Delta s, \quad t \in T_0, \tag{2.1}$$

then

$$u(t) \le G^{-1}\Big[G(a(t)) + \int_{t_0}^{t} f(t,s)\Delta s\Big], \quad t \in T_0, \tag{2.2}$$

where

$$G(v) = \int_{v_0}^{v} \frac{1}{\omega(r)}\mathrm{d}r, \quad v \ge v_0 > 0,\ G(+\infty) = +\infty. \tag{2.3}$$

Proof For arbitrarily fixed $\widetilde{t_0} > t_0$, by the condition, we have

$$u(t) \le a(\widetilde{t_0}) + \int_{t_0}^{t} f(\widetilde{t_0}, s)\omega(u(s))\Delta s, \quad t \in [t_0, \widetilde{t_0}].$$

Let $z(t) = a(\widetilde{t_0}) + \int_{t_0}^{t} f(\widetilde{t_0}, s)\omega(u(s))\Delta s$, then we get $z(t_0) = a(\widetilde{t_0})$ and $u(t) \le z(t)$. Since

$$z^{\Delta}(t) = f(\widetilde{t_0}, t)\omega(u(t)) \le f(\widetilde{t_0}, t)\omega(z(t)),$$

we have

$$\frac{z^{\Delta}(t)}{\omega(z(t))} \le f(\widetilde{t_0}, t).$$

Furthermore, for $t \in [t_0, \widetilde{t_0}]$, if $\sigma(t) > t$, then

$$\begin{aligned}[G(z(t))]^{\Delta} &= \frac{G(z(\sigma(t))) - G(z(t))}{\sigma(t) - t} = \frac{1}{\sigma(t) - t}\int_{z(t)}^{z(\sigma(t))} \frac{1}{\omega(r)}\mathrm{d}r \\ &\le \frac{z(\sigma(t)) - z(t)}{\sigma(t) - t}\frac{1}{\omega(z(t))} = \frac{z^{\Delta}(t)}{\omega(z(t))}.\end{aligned}$$

If $\sigma(t) = t$, then