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A CLASS OF SPECTRALLY ARBITRARY RAY PATTERNS *

Jiangwu Deng†

(*College of Math. and Computer Science, Fuzhou University, Fuzhou 350116, Fujian, PR China*)

Abstract

An $n \times n$ ray pattern A is said to be spectrally arbitrary if for every monic nth degree polynomial $f(x)$ with coefficients from $\mathbb{C}$, there is a complex matrix in the ray pattern class of A such that its characteristic polynomial is $f(x)$. In this paper, a family ray patterns is proved to be spectrally arbitrary by using Nilpotent-Jacobian method.

Keywords ray pattern; Nilpotent-Jacobian method; spectrally arbitrary
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1 Introduction

A *ray pattern* $A = (a_{jk})$ of order n is a matrix with entries $a_{jk} \in \{e^{i\theta} | \, 0 \leq \theta < 2\pi\} \cup \{0\}$, where $i^2 = -1$. Its ray pattern class is

$$Q_R(A) = \{B = (b_{jk}) \in M_n(\mathbb{C}) | \, b_{jk} = r_{jk}a_{jk}, \, r_{jk} \in \mathbb{R}^+, \, 1 \leq j, k \leq n\}.$$

It is easy to see that ray patterns are a generalization of the sign patterns.

A ray pattern A is said to be *spectrally arbitrary* if for any monic nth degree polynomial $f(x)$ with coefficients from $\mathbb{C}$, there is a complex matrix $B \in Q_R(A)$ such that the characteristic polynomial of B is $f(x)$.

Spectrally arbitrary problem is a basic subject in combinatorial matrix theory and a hot topic for some international scholars. The problem of the spectrally arbitrary sign patterns was introduced in [2]. J.H. Drew et al. developed the Nilpotent-Jacobian method to show that a sign pattern is spectrally arbitrary in [2]. Work on spectrally arbitrary sign patterns has continued in several articles including [1,3,4]. J.J. Mcdonald and J. Stuart in [6] extended the Nilpotent-Jacobian method from sign patterns to the ray patterns. Y.Z. Mei and Y.B. Gao in [7] showed that the minimum number of nonzeros in an $n \times n$ irreducible spectrally arbitrary ray pattern is $3n - 1$.

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†Corresponding author. E-mail: 870531288@qq.com

Though the general method — Nilpotent-Jacobian method — to prove the spectrally arbitrary property has been developed, the proof procedure is not very easy. Let $A_{n,m} = (a_{jk})$ be an $n \times n$ complex square matrix as follows

$$
\begin{array}{c} \\ \\ \\ \\ \\ \\ A_{n,m}= \\ \\ \\ \\ \\ \\ \end{array}
\begin{array}{c} \\ 1 \\ \\ \vdots \\ \\ m \\ \\ \\ \\ \vdots \\ \\ \\ n \end{array}
\begin{array}{cccccccccccc}
1 & & \cdots & & m & & & \cdots & & & & n \\
\left(\begin{array}{c}-1\end{array}\right. & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
-1 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
1 & 0 & 0 & 0 & e^{i\theta} & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 \\
1 & -i & -i & -i & \cdots & \cdots & \cdots & \cdots & \cdots & -i & -i & \left.-i\right)
\end{array}
(0 \le \theta < 2\pi),
$$

where $n,\ m,\ j$ and k are positive integers; $2 \le m \le n-2$, $1 \le j,k \le n$; and the (m,m) entry is $e^{i\theta}$.

In [6], the ray pattern $A_{n,2}$ was proved to be spectrally arbitrary. In [8], the ray pattern $A_{n,3}$ was proved to be spectrally arbitrary. In [5,9], several families ray patterns were proved to be spectrally arbitrary.

In this paper, we show that for $n \ge 8$ if $\theta \in \left(\arccos\frac{2}{\sqrt{5}}, \arccos\sqrt{\frac{3+\sqrt{3}}{6}}\right)$, then the ray pattern $A_{n,4}$ is spectrally arbitrary.

2 The Extended Nilpotent-Jacobian Method

A square matrix A is called to be *nilpotent* if there exists a positive integer k such that $A^k = 0$ but $A^{k-1} \ne 0$. A ray pattern B is said to be *potentially nilpotent* if there is a complex matrix $A \in Q_R(B)$ with characteristic polynomial $g(x) = x^n$. If the ray pattern A is spectrally arbitrary, then A is potentially nilpotent affirmatively.

In [6], the extended Nilpotent-Jacobian method can be summarized as follows:

(1) Find a nilpotent matrix in the given ray pattern class.

(2) Change $2n$ of the positive coefficients (denoted $r_1, r_2, \cdots, r_{2n}$) of the $e^{i\theta_{jk}}$ in this nilpotent matrix to variables $t_1, t_2, \cdots, t_{2n}$.

(3) Express the characteristic polynomial of the resulting matrix as:

$$x^n + \sum_{k=1}^{n}(f_k(t_1, t_2, \cdots, t_{2n}) + ig_k(t_1, t_2, \cdots, t_{2n}))x^{n-k}.$$