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Well-Posedness of the Free Boundary Problem for the Compressible Euler Equations and the Incompressible Limit

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Abstract. In this paper, we study the free boundary problem of the compressible Euler equations in the Eulerian coordinates. By deriving the evolution equation of the free surface, we relate the Taylor stability condition to the hyperbolicity of this evolution equation. Our approach not only yields exact information of the free surface, but also gives a simple proof of the local well-posedness of the free boundary problem. This approach provides a unified framework to treat both compressible and incompressible free boundary problems. As a byproduct, we can also prove the incompressible limit.

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1 Introduction

1.1 Presentation of the problem

The compressible Euler equations are

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$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0, \\ \partial_t u + u \cdot \nabla u + \dfrac{\nabla p}{\rho} = 0, \end{cases} \tag{1.1}$$

where ρ is the density and u is the velocity of a compressible liquid. The pressure p is given by the state equation

$$p = p(\rho) = \frac{1}{\epsilon^2}(\rho - 1) \tag{1.2}$$

with $0 < \epsilon < 1$ as the inverse of the sound speed. The method here also works for more general state equations, but here we choose the linear one (1.2) to simplify our arguments.

We are considering the free boundary problem (FBP) in the domain

$$\Omega_t = \left\{ x = (x^1, x^2, x^3) = (\overline{x}, x^3) : \overline{x} \in \mathbb{T}^2, -1 < x^3 < f(t, \overline{x}) \right\}$$

with the free boundary given by a two dimensional surface

$$\Gamma_t = \left\{ (\overline{x}, x^3) : \overline{x} \in \mathbb{T}^2, x^3 = f(t, \overline{x}) \right\}.$$

On the free surface, there holds that

$$\begin{cases} u \cdot n = V(t, \overline{x}), \\ p = 0, \end{cases} \tag{1.3}$$

where V is the normal velocity of Γ_t and n is the unit outer normal direction of Γ_t. Since the free surface separates the fluid and the vacuum, we have the evolution equation of the free surface

$$\partial_t f = u \cdot N \tag{1.4}$$

with $N = (-\partial_1 f, -\partial_2 f, 1)^\top$ and $n = \frac{N}{|N|}$. On the bottom of the domain $\Gamma_- = \{(\overline{x}, -1) : \overline{x} \in \mathbb{T}^2\}$, we pose the slip boundary condition

$$u^3 = 0 \quad \text{on } \Gamma_-. \tag{1.5}$$

The initial data in

$$\Omega_0 = \left\{ (\overline{x}, x^3) : \overline{x} \in \mathbb{T}^2, -1 < x^3 < f_{\text{in}} \right\}$$

are given by

$$\rho(0, x) = \rho_{\text{in}}(x), \quad u(0, x) = u_{\text{in}}(x). \tag{1.6}$$