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Definite Condition of the Evolutionary $\vec{p}(x)-$Laplacian Equation

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Abstract. For the nonlinear degenerate parabolic equations, how to find an appropriate boundary value condition to ensure the well-posedness of weak solution has been an interesting and challenging problem. In this paper, we develop the general characteristic function method to study the stability of weak solutions based on a partial boundary value condition.

Key Words: Definite condition, stability, general characteristic function method, weak solution, Laplacian equation.

AMS Subject Classifications: 35B35, 35D30, 35K55

1 Introduction

Consider the evolutionary $\vec{p}(x)-$Laplacian equation [20]

$$\begin{aligned} u_t = &\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)\,|u_{x_i}|^{p_i(x)-2}\,u_{x_i}\right) + \sum_{i=1}^{N} \frac{\partial b_i(u,x,t)}{\partial x_i} \\ &- b(x,t)|u|^{\sigma(x)-2}u, \quad (x,t)\in\Omega\times(0,T), \end{aligned} \tag{1.1}$$

where $a_i(x)$, $p_i(x)$ and $\sigma(x)$ are nonnegative continuous functions with $p_i(x) > 1$ and $\sigma(x) > 1$, $b(x,t)$ and $b_i(s,x,t)$ are Lipschitz functions, and Ω is a smooth bounded domain in $\mathbb{R}^n$ with $\Omega_T = \Omega\times(0,T)$, $T\in(0,\infty)$. A simpler version of Eq. (1.1) is of the form

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)|u_{x_i}|^{p_i(x)-2}u_{x_i}\right), \tag{1.2}$$

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which is the so-called anisotropic electrorheological fluid equation [1,17]. When $a_i(x) \equiv 1$, the Cauchy-Dirichlet problem and the Cauchy problem for the degenerate and singular quasilinear anisotropic parabolic equations were studied in [13,18,19]. If $a_i(x) \in C^1(\overline{\Omega})$ satisfies

$$a_i(x) > 0, \quad x \in \Omega \quad \text{and} \quad a_i(x) = 0, \quad x \in \partial\Omega, \tag{1.3}$$

the well-posedness of Eq. (1.2) was established in [21]. The degenerate parabolic p-Laplace equation with measurable coefficients was investigated in [6] and the improved integrability of the gradient was naturally formulated in terms of Marcinkiewicz spaces.

Antontsev-Shmarev [3] considered the existence of weak solution of the equation

$$u_t = \operatorname{div}\left(a(x,t)\,|\nabla u|^{p(x)-2}\,\nabla u\right) - b(x,t)|u|^{\sigma(x)-2}u, \quad (x,t) \in \Omega \times (0,T),$$

and investigated the vanishing property of solutions under the suitable assumptions on $b(x,t)$ and the variable exponent $\sigma(x)$ [4]. Chen-Perthame [8] studied the well-posedness and stability of a class of nonlinear hyperbolic-parabolic equations by developing an analytical and effective approach. Recently, we studied the well-posedness of an anisotropic parabolic equation [22]

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)\,|u_{x_i}|^{p_i-2}\,u_{x_i}\right) + f(x,t,u,\nabla u), \quad (x,t) \in \Omega \times (0,T).$$

When some diffusion coefficients are degenerate on the boundary $\partial\Omega$ and the others are positive on $\overline{\Omega}$, a new concept–the general characteristic function of the domain Ω, was introduced and applied, and a novel partial boundary value condition was presented to study the stability of weak solutions for anisotropic parabolic equations.

Distinguished from those [21,22] in which $a_i(x)$ is requested to satisfy condition (1.3), in this paper we consider the well-posedness of weak solutions to Eq. (1.1) by only requiring $a_i(x) \geq 0$, $i = 1,2,\cdots,N$ and

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.4a}$$

$$u(x,t) = 0, \qquad (x,t) \in \Sigma_p \times (0,T), \tag{1.4b}$$

where $\Sigma_p \subset \partial\Omega$ is a relatively open subset.

For the associated linear case of Eq. (1.1), i.e., the degenerate linear heat conduction equation of the form

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)u_{x_i}\right) + \sum_{i=1}^{N} b^i(x)u_{x_i} + b(x,t)u + g(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1.5}$$

where $a_i(x) = 0$ on the boundary $\partial\Omega$, to ensure the well-posedness and stability of weak solution, according to the Fichera-Oleinik theory [10,16], we need to include a partial boundary condition as (1.4b), in which

$$\Sigma_p = \left\{x \in \partial\Omega : \sum_{i=1}^{N} b^i(x)n_i(x) < 0\right\}, \tag{1.6}$$