Ann. Appl. Math.
doi: 10.4208/aam.OA-2022-0008

Vol. **38**, No. 3, pp. 356-384
August 2022

Negligible Obstructions and Turán Exponents

Tao Jiang[1], Zilin Jiang[2,*] and Jie Ma[3]

[1] *Department of Mathematics, Miami Univeristy, Oxford, OH 45056, USA*
[2] *School of Mathematical and Statistical Sciences, and School of Computing and Augmented Intelligence, Arizona State University, Tempe, AZ 85281, USA*
[3] *School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China*

Received 31 March 2022; Accepted (in revised version) 31 July 2022

Abstract. We show that for every rational number $r\in(1,2)$ of the form $2-a/b$, where $a,b\in\mathbb{N}^+$ satisfy

$$\lfloor b/a\rfloor^3\le a\le b/(\lfloor b/a\rfloor+1)+1,$$

there exists a graph F_r such that the Turán number $\mathrm{ex}(n,F_r)=\Theta(n^r)$. Our result in particular generates infinitely many new Turán exponents. As a byproduct, we formulate a framework that is taking shape in recent work on the Bukh–Conlon conjecture.

AMS subject classifications: 05C35

Key words: Extremal graph theory, turán exponents, bipartite graphs.

1 Introduction

Given a family $\mathcal{F}$ of graphs, the Turán number $\mathrm{ex}(n,\mathcal{F})$ is defined to be the maximum number of edges in a graph on n vertices that contains no graph from the family $\mathcal{F}$ as

*Corresponding author.
Emails: `jiangt@miamioh.edu` (T. Jiang), `zilinj@asu.edu` (Z. Jiang), `jiema@ustc.edu.cn` (J. Ma)

a subgraph. The classical Erdős–Stone–Simonovits theorem shows that arguably the most interesting problems about Turán numbers, known as the degenerate extremal graph problems, are to determine the order of magnitude of $\mathrm{ex}(n,\mathcal{F})$ when $\mathcal{F}$ contains a bipartite graph. The following conjecture attributed to Erdős and Simonovits is central to Degenerate Extremal Graph Theory (see [16, Conjecture 1.6]).

Conjecture 1.1 (Rational Exponents Conjecture)**.** For every finite family $\mathcal{F}$ of graphs, if $\mathcal{F}$ contains a bipartite graph, then there exists a rational $r\in[1,2)$ and a positive constant c such that $\mathrm{ex}(n,\mathcal{F})=cn^r+o(n^r)$.

Recently Bukh and Conlon made a breakthrough on the inverse problem [16, Conjecture 2.37].

Theorem 1.1 (Bukh and Conlon [3])**.** *For every rational number $r\in(1,2)$, there exists a finite family of graphs $\mathcal{F}_r$ such that $\mathrm{ex}(n,\mathcal{F}_r)=\Theta(n^r)$.*

Motivated by another outstanding problem of Erdős and Simonovits (see [10, Section III] and [11, Problem 8]), subsequent work has been focused on the following conjecture, which aims to narrow the family $\mathcal{F}_r$ in Theorem 1.1 down to a single graph.

Conjecture 1.2 (Realizability of Rational Exponents)**.** For every rational number $r\in(1,2)$, there exists a bipartite graph F_r such that $\mathrm{ex}(n,F_r)=\Theta(n^r)$.†

It is believed that the graph F_r in Conjecture 1.2 could be taken from a specific yet rich family of graphs, for which we give the following definitions.

Definition 1.1. *A rooted graph is a graph F equipped with a subset $R(F)$ of vertices, which we refer to as roots. We define the pth power of F, denoted F^p, by taking the disjoint union of p copies of F, and then identifying each root in $R(F)$, reducing multiple edges (if any) between the roots.*

Definition 1.2. *Given a rooted graph F, we define the density ρ_F of F to be $e(F)/(v(F)-|R(F)|)$, where $v(F)$ and $e(F)$ denote the number of vertices and respectively edges of F. We say that a rooted graph F is balanced if $\rho_F>1$, and for every subset S of $V(F)\setminus R(F)$, the number of edges in F with at least one endpoint in S is at least $\rho_F|S|$.*

Indeed the next result on Turán numbers, which follows immediately from [3, Lemma 1.2], establishes the lower bound in Conjecture 1.2 for some power of a balanced rooted tree.‡

---

†Erdős and Simonovits asked a much stronger question: for every rational number $r\in(1,2)$, find a bipartite graph F_r such that $\mathrm{ex}(n,F_r)=cn^r+o(n^r)$ for some positive constant c.

‡A rooted tree is a rooted graph that is also a tree, not to be confused with a tree having a designated vertex.