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Hamilton-Jacobi Equations for Nonholonomic Magnetic Hamiltonian Systems

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In Memory of Great Geometer Shiing Shen Chern

Abstract. In order to describe the impact of the different geometric structures and the constraints for the dynamics of a Hamiltonian system, in this paper, for a magnetic Hamiltonian system defined by a magnetic symplectic form, we drive precisely the geometric constraint conditions of the magnetic symplectic form for the magnetic Hamiltonian vector field, which are called the Type I and Type II Hamilton-Jacobi equations. Second, for the magnetic Hamiltonian system with a nonholonomic constraint, we can define a distributional magnetic Hamiltonian system, then derive its two types of Hamilton-Jacobi equations. Moreover, we generalize the above results to nonholonomic reducible magnetic Hamiltonian system with symmetry, we define a nonholonomic reduced distributional magnetic Hamiltonian system, and prove the two types of Hamilton-Jacobi theorems. These research reveal the deeply internal relationships of the magnetic symplectic structure, the nonholonomic constraint, the distributional two-form, and the dynamical vector field of the nonholonomic magnetic Hamiltonian system.

AMS subject classifications: 70H20, 70F25, 53D20

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1 Introduction

It is well-known that Hamilton-Jacobi theory is an important research subject in

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mathematics and analytical mechanics (see Abraham and Marsden [1], Arnold [2] and Marsden and Ratiu [19]), and the Hamilton-Jacobi equation is also fundamental in the study of the quantum-classical relationship in quantization, and it plays an important role in the study of stochastic dynamical systems (see Woodhouse [32], Ge and Marsden [10], and Lázaro-Camí and Ortega [12]). For these reasons, the equation is described as a useful tool in the study of Hamiltonian system theory, which has been extensively developed in recent years and become one of the most active subjects in the study of modern applied mathematics and analytical mechanics.

The Hamilton-Jacobi theory, from the variational point of view, was originally developed by Jacobi in 1866, and it states that the integral of the Lagrangian of a mechanical system along the solution of its Euler-Lagrange equation satisfies the Hamilton-Jacobi equation. The classical description of this problem, from the generating function and the geometrical point of view, was given by Abraham and Marsden in [1] as follows: letting Q be a smooth manifold and TQ the tangent bundle, T^*Q is the cotangent bundle with a canonical symplectic form ω, and the projection $\pi_Q: T^*Q \to Q$ induces the map $T\pi_Q: TT^*Q \to TQ$.

Theorem 1.1. *Assume that the triple (T^*Q,ω,H) is a Hamiltonian system with Hamiltonian vector field X_H, and $W: Q \to \mathbb{R}$ is a given generating function. Then the following two assertions are equivalent:*

- *(i) For every curve $\sigma: \mathbb{R} \to Q$ satisfying that $\dot{\sigma}(t) = T\pi_Q(X_H(\mathbf{d}W(\sigma(t))))$, $\forall t \in \mathbb{R}$, then $\mathbf{d}W \cdot \sigma$ is an integral curve of the Hamiltonian vector field X_H.*
- *(ii) W satisfies the Hamilton-Jacobi equation $H(q^i, \frac{\partial W}{\partial q^i}) = E$, where E is a constant.*

From the proof of the above theorem given in Abraham and Marsden [1], we know that the assertion (i), equivalent to Hamilton-Jacobi equation (ii) by the generating function, gives a geometric constraint condition of the canonical symplectic form on the cotangent bundle T^*Q for the Hamiltonian vector field of the system. Thus, the Hamilton-Jacobi equation reveals the deeply internal relationships of the generating function, the canonical symplectic form, and the dynamical vector field of a Hamiltonian system.

Now, it is a natural problem how to generalize Theorem 1.1 to fit the nonholonomic systems and their reduced systems. Note that if we take that $\gamma = \mathbf{d}W$ in Theorem 1.1, then γ is a closed one-form on Q, and the equation $\mathbf{d}(H \cdot \mathbf{d}W) = 0$ is equivalent to the Hamilton-Jacobi equation $H(q^i, \frac{\partial W}{\partial q^i}) = E$, where E is a constant, which was called the classical Hamilton-Jacobi equation. This result was used the formulation of a geometric version of the Hamilton-Jacobi theorem for