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Asymptotic Behavior of Solutions to a Class of Semilinear Parabolic Equations with Boundary Degeneracy

Xinxin Jing, Chunpeng Wang* and Mingjun Zhou

School of Mathematics, Jilin University, Changchun 130012, China.

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Abstract. This paper concerns the asymptotic behavior of solutions to one-dimensional semilinear parabolic equations with boundary degeneracy both in bounded and unbounded intervals. For the problem in a bounded interval, it is shown that there exist both nontrivial global solutions for small initial data and blowing-up solutions for large one if the degeneracy is not strong. Whereas in the case that the degeneracy is strong enough, the nontrivial solution must blow up in a finite time. For the problem in an unbounded interval, blowing-up theorems of Fujita type are established. It is shown that the critical Fujita exponent depends on the degeneracy of the equation and the asymptotic behavior of the diffusion coefficient at infinity, and it may be equal to one or infinity. Furthermore, the critical case is proved to belong to the blowing-up case.

AMS subject classifications: 35K65, 35D30, 35B33

Key words: Asymptotic behavior, boundary degeneracy, blowing-up.

1 Introduction

In this paper, we consider the following semilinear degenerate equation of the form:

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)=f(x,t,u),\quad 0<x<1,\quad t>0,\tag{1.1}$$

*Corresponding author. *Email address:* `wangcp@jlu.edu.cn` (C. Wang)

where $a\in C([0,1])\cap C^1((0,1])$ such that $a>0$ in $(0,1]$ and $a(0)=0$. As a parabolic equation with boundary degeneracy, (1.1) is degenerate at $x=0$, a portion of the lateral boundary. Such equations are used to describe some models, such as the Budyko-Sellers climate model [18], the Black-Scholes model coming from the option pricing problem [3], and a simplified Crocco-type equation coming from the study on the velocity field of a laminar flow on a flat plate [7]. The typical case of a is

$$a(x)=x^\lambda, \quad x\in[0,1], \quad \lambda>0. \tag{1.2}$$

In recent years, the null controllability of the control system governed by (1.1) was studied in [1, 8, 9, 17, 22, 25, 26]. In particular, the following control system was studied:

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(x^\lambda\frac{\partial u}{\partial x}\right)+c(x,t)u=h(x,t)\chi_\omega, \qquad (x,t)\in(0,1)\times(0,T), \tag{1.3}$$

$$\begin{cases} u(0,t)=u(1,t)=0, & \text{if} \quad 0<\lambda<1, \\ \lim\limits_{x\to0^+}x^\lambda\dfrac{\partial u}{\partial x}(x,t)=u(1,t)=0, & \text{if} \quad \lambda\geq1, \end{cases} \qquad t\in(0,T), \tag{1.4}$$

$$u(x,0)=u_0(x), \qquad x\in(0,1), \tag{1.5}$$

where $\lambda>0, c\in L^\infty((0,1)\times(0,T))$. It was shown that the system (1.3)-(1.5) is null controllable if $0<\lambda<2$, while not if $\lambda\geq2$. Although the system (1.3)-(1.5) is not null controllable for $\lambda\geq2$, it was proved in [11, 19, 21] and [4–6] that it is approximately controllable in $L^2((0,1))$ and regional null controllable for each $\lambda>0$, respectively.

In this paper, we study the asymptotic behavior of solutions to (1.1) with

$$f(x,t,u)=u^p, \quad (x,t,u)\in(0,1)\times(0,+\infty)\times\mathbb{R}, \quad p>1.$$

That is to say, we consider the following problem:

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)=u^p, \qquad (x,t)\in(0,1)\times(0,T), \tag{1.6}$$

$$\lim_{x\to0^+}a(x)\frac{\partial u}{\partial x}(x,t)=0, \;\; u(1,t)=0, \quad t\in(0,T), \tag{1.7}$$

$$u(x,0)=u_0(x), \qquad x\in(0,1). \tag{1.8}$$

By a weighted energy estimate, it is shown that the asymptotic behavior of solutions to the problem (1.6)-(1.8) depends on the degenerate rate of a at $x=0$. Precisely, it is assumed that $a\in C([0,1])\cap C^1((0,1])$ satisfies

$$a(0)=0, \quad a(x)>0 \quad \text{for} \quad 0<x\leq1. \tag{1.9}$$