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# Regularity for 3-D MHD Equations in Lorentz Space $L^{3,\infty}$

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**Abstract.** The regularity for 3-D MHD equations is considered in this paper, it is proved that the solutions $(v,B,p)$ are Hölder continuous if the velocity field $v\in L^\infty(0,T;L_x^{3,\infty}(\mathbb{R}^3))$ with local small condition

$$r^{-3}\left|\left\{x\in B_r(x_0):|v(x,t_0)|>\varepsilon r^{-1}\right\}\right|\le\varepsilon$$

and the magnetic field $B\in L^\infty(0,T;VMO^{-1}(\mathbb{R}^3))$.

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**Key words**: Lorentz space, backward uniqueness, MHD equations.

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## 1 Introduction

In this paper, we will consider the following MHD equations in $\mathbb{R}^3\times(0,T)$:

$$\begin{cases} \partial_t v+v\cdot\nabla v-\Delta v+\nabla p=B\cdot\nabla B, \\ \partial_t B-\Delta B+v\cdot\nabla B-B\cdot\nabla v=0, \\ \mathrm{div} v=0, \quad \mathrm{div} B=0, \end{cases} \tag{1.1}$$

where $v$ is the fluid velocity field and $B$ is the magnetic field. The function $p$ describes the scalar pressure. The MHD equations (1.1) reduce to the incompressible Navier-Stokes equations when there is no electromagnetic field, that is, $B\equiv 0$.

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There is a very rich literatures dedicated to the mathematical study of the Navier-Stokes system by many researchers; see, for example, [1,3–5,8–10,13–15,18,20,22]. Compared with Navier-Stokes equation, the MHD equation (1.1) is a combination of the incompressible Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. Such a system depicts the motion of many conducting impressible immiscible fluids without surface tension under the action of a magnetic field.

For a point $z=(x,t)\in\mathbb{R}^3\times\mathbb{R}_+$, we denote

$$B_r(x):=\left\{y\in\mathbb{R}^3:|y-x|<r\right\},\quad Q_r(z):=B_r(x)\times\left(t-r^2,t\right),\quad Q_r=Q_r(0,0),$$

and define the space

$$L^{q,p}(Q_r)=L^q\left(t-r^2,t;L^p\left(B_r(x)\right)\right)$$

with the norm

$$\|u\|^q_{L^{p,q}(Q_r(z))}=\int_{t-r^2}^t\|u(t)\|^q_{L^p(B_r(x))}dt.$$

Let $\Omega\subset\mathbb{R}^3$. The Lorentz space $L^{(p,q)}(\Omega)$ with $p,q\in(0,\infty)$ is the set of measurable functions $f$ on $\Omega$ such that the following quasi-norm is bounded:

$$\|f\|_{L^{(p,q)}(\Omega)}:=\begin{cases}\left(p\displaystyle\int_0^\infty\alpha^q\left|\{x\in\Omega:|f(x)|>\alpha\}\right|^{\frac{q}{p}}\frac{d\alpha}{\alpha}\right)^{\frac{1}{q}}, & \text{if}\quad q<\infty,\\ \sup_{\alpha>0}\alpha\left|\{x\in\Omega:|f(x)|>\alpha\}\right|^{\frac{1}{p}}, & \text{if}\quad q=\infty.\end{cases}$$

Lorentz spaces can be used to capture logarithmic singularities. For example, in $\mathbb{R}^3$, for any $\beta>0$ we have

$$|x|^{-1}\left(\frac{\log|x|}{2}\right)^{-\beta}\in L^{(3,q)}\left(\mathbb{R}^3\right),\quad \text{if and if}\quad q>\frac{1}{\beta}.$$

It is known that $L^{(p,p)}(\Omega)=L^p(\Omega)$ and $L^{(p,q_1)}(\Omega)\subset L^{(p,q_2)}(\Omega)$ while $q_1\le q_2$. If $|\Omega|$ is finite then $L^{(p,q)}(\Omega)\subset L^r(\Omega)$ for all $0<q\le\infty$ and $0<r<p$,

$$\|g\|_{L^r(\Omega)}\le|\Omega|^{\frac{1}{r}-\frac{1}{p}}\|g\|_{L^{(p,q)}(\Omega)}. \tag{1.2}$$

In the classical literatures, Leray [9] and Hopf [8] established the existence weak solution to the Navier Stokes system. However, the smoothness of the Leray-Hopf weak solution is still a challenging open problem. Serrin [18] proved