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A Robust Hybrid Spectral Method for Nonlocal Problems with Weakly Singular Kernels

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Abstract. In this paper, we propose a hybrid spectral method for a type of nonlocal problems, nonlinear Volterra integral equations (VIEs) of the second kind. The main idea is to use the shifted generalized Log orthogonal functions (GLOFs) as the basis for the first interval and employ the classical shifted Legendre polynomials for other subintervals. This method is robust for VIEs with weakly singular kernel due to the GLOFs can efficiently approximate one-point singular functions as well as smooth functions. The well-posedness and the related error estimates will be provided. Abundant numerical experiments will verify the theoretical results and show the high-efficiency of the new hybrid spectral method.

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Key words: Nonlocal problem, Volterra integral, spectral element method, log orthogonal function, Legendre polynomial, weak singularity, exponential convergence.

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1. Introduction

In model reduction, nonlocality arises naturally. Based on the wide application of multiscale and stochastic modeling in various fields such as materials science, thermodynamics, image analysis fluid dynamics and fracture mechanics [1, 2, 17], nonlocal modeling plays an important role and develops rapidly. Therefore, many scholars have done more in-depth research on nonlocal problems, including the application of various nonlocal equations [11, 13–16, 31]. As a well known case, the integral equation is

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a very important branch of mathematics. According to the number of unknown functions, integral equations can be divided into first-kind, second-type and third-type. This paper is concerned with the following nonlinear second-kind VIEs:

$$y(t)=f(t)+\int_0^t (t-s)^{-\mu}K(t,s)G\big(s,y(s)\big)\mathrm{d}s, \quad t\in U=[0,T], \tag{1.1}$$

where $0<\mu<1$, $K\in C(D)$, $D:=\{(t,s):0\le s\le t\le T\}$, $f\in C(U)$ and G is a continuous function of s and y.

There are a wide range of applications modeled by VIEs arising in physics, biology and other fields. Originally, Volterra oneself considered the numerical solution of integral equation in his book [35] and references therein. Afterwards, Chen *et al.* [6] obtained the error estimation by using the finite element method in space and the backward Euler scheme in time direction. Besides, piecewise polynomial collocation methods and Runge-Kutta methods [3, 4, 12, 34, 37] also have many analysis on VIEs. However, most local methods may be unsuitable for the nonlocal problems. In recent decades, spectral methods have been developed rapidly and widely used in various fields [5, 18, 20, 21, 26] and references therein. Spectral methods provide exceedingly accurate numerical results with relatively fewer degrees of freedom for smooth solutions. Therefore it has been widely used for VIEs with smooth kernel and solutions [19, 29, 30, 32, 36, 38]. However, the regularity of the solutions is low for VIEs with weakly singular kernel. The traditional spectral methods based on polynomials cannot obtain the efficient approximation to the corresponding solution.

In order to overcome the difficulties caused by the weakly singular kernel and enhance the convergence rate, abundant researchers made a lot of efforts for handling the singularity. Feldstem *et al.* [23,33] analyzed the most fundamental problem. Pedas and Vainikko [24] used piecewise polynomial collocation method through smooth transformation to deal with weakly singular kernels. After that, Chen and Tang [10] analyzed the convergence of the Jacobi spectral-collocation methods for VIEs with weakly singular kernel. Hou *et al.* [22] use Müntz spectral methods to deal with the weakly singular kernel. Since the solution of the weakly singular VIEs (1.1) is singular only at $t=0$ but smooth for $t\neq 0$, we need to deal with the singularity in the first interval. Sheng and Shen proposed a hybrid spectral element method with mixed generalized Jacobi function [8,25] and Legendre polynomial to solve weakly kernel VIEs [28] where they divided the original domain into some subintervals and use the generalized Jacobi functions to approximate the solution in the first subinterval. The method proposed by Sheng and Shen performed excellent for a special case but not for the more general cases.

In this paper, we propose a universal method by replacing the basis in the first interval by GLOFs [7,9] and use a new hybrid spectral element method to solve VIEs (1.1). The main contributions and advantages of the new method highlighted as follows:

- Our hybrid spectral method can exponentially approximate the singular solutions caused by the weakly singular kernel in (1.1) as well as the smooth solutions.