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Avoiding Small Denominator Problems by Means of the Homotopy Analysis Method

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Abstract. The so-called "small denominator problem" was a fundamental problem of dynamics, as pointed out by Poincaré. Small denominators appear most commonly in perturbative theory. The Duffing equation is the simplest example of a non-integrable system exhibiting all problems due to small denominators. In this paper, using the forced Duffing equation as an example, we illustrate that the famous "small denominator problems" never appear if a non-perturbative approach based on the homotopy analysis method (HAM), namely "the method of directly defining inverse mapping" (MDDiM), is used. The HAM-based MDDiM provides us great freedom to directly define the inverse operator of an undetermined linear operator so that all small denominators can be completely avoided and besides the convergent series of multiple limit-cycles of the forced Duffing equation with high nonlinearity are successfully obtained. So, from the viewpoint of the HAM, the famous "small denominator problems" are only artifacts of perturbation methods. Therefore, completely abandoning perturbation methods but using the HAM-based MDDiM, one would be never troubled by "small denominators". The HAM-based MDDiM has general meanings in mathematics and thus can be used to attack many open problems related to the so-called "small denominators".

AMS subject classifications: 41A58, 34C25

Key words: Small denominator problem, Duffing equation, limit cycle, homotopy analysis method (HAM), MDDiM.

1 Origin of "small denominator problem"

Poincaré [1] pointed out that the so-called "small denominator problem" was "the fundamental problem of dynamics". The small denominator was first mentioned by Delaunay [2] in his 900 pages book about celestial motions using perturbation method.

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Poincaré [1] first recognized that, when small denominator appears, the coefficients of perturbation series may grow too large too often, threatening the convergence of the series. As pointed out by Pérez [3], "small denominators are found most commonly in the perturbative theory". It often appears when perturbation methods are used to solve problems in classical and celestial mechanics [4], fluid mechanics [5,6], and so on [7,8].

What is the origin of the so-called "small denominator problem"? As pointed out by Giorgilli [9], the Duffing equation [10] "is perhaps the simplest example of a non-integrable system exhibiting **all** problems due to the small denominators". So, without loss of generality, let us focus on the forced Duffing equation

$$\mathcal{N}[u(t)] = u''(t) + 2\xi u'(t) + u(t) + \beta u^3(t) - \alpha\cos(\Omega t) = 0, \tag{1.1}$$

where $\mathcal{N}$ is a nonlinear operator, the prime denotes the differentiation with respect to the time t, α and Ω is the amplitude and frequency of the external force $F = \alpha\cos(\Omega t)$, $\xi > 0$ is the resistance coefficient, and $\beta > 0$ is a physical parameter related to nonlinearity, respectively.

As pointed out by Kartashova [11], "physical classification of PDEs is based not on the form of equations, but on the form of solutions". So, let us consider here the stationary periodic limit-cycle of $u(t)$ as $t \to +\infty$ of the forced Duffing equation (1.1), which can be expressed in the form:

$$u(t) = \sum_{n=1}^{+\infty} \Big\{ a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \Big\}, \tag{1.2}$$

where a_n, b_n are constants and

$$\omega_n = (2n-1)\Omega, \quad n \geq 1. \tag{1.3}$$

This is mainly because the common solution

$$A\exp(-\xi t)\cos(t) + B\exp(-\xi t)\sin(t)$$

of the linear equation

$$u''(t) + 2\xi u'(t) + u(t) = 0$$

tends to zero as $t \to +\infty$ for arbitrary constants A and B, and thus disappear in the so-called "solution-expression" (1.2) of the limit-cycle.

Let us first show how perturbation technique [12, 13] can bring the so-called small denominators into the above-mentioned problem. Let β be a small parameter and assume that $u(t)$ can be expanded in such a series

$$u(t) = u_0(t) + \sum_{n=1}^{+\infty} u_n(t)\beta^n. \tag{1.4}$$